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THE GENERALIZED COINCIDENCE INDEX – APPLICATION TO A BOUNDARY VALUE PROBLEM

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Abstract. In this paper we investigate a general boundary value problem, which can be rewritten to the coincidence problem of the form $L(x) = F(x)$, where L is a Fredholm operator of nonnegative index and F is not necessarily compact map. We apply a homotopy invariant called a coincidence index.

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1. Introduction

Let $AC = AC([0,T],E)$ be the space of absolutely continuous functions $u : [0,T] \to E$ defined on the unit interval $[0,T]$ with values in a Banach space E and let $f : [0,1] \times E \times E \to E$ be a Caratheodory map, what means that $f(\cdot, u, v)$ is mesurable for every $(u,v) \in E \times E$ and $f(t,\cdot,\cdot)$ is continuous for a.a. $t \in [0,T]$. If we are to study the existence of solutions to the general boundary value problem

$$\begin{cases} u'(t) = f(t, u(t), u'(t)) \\ l_1(u(0)) + l_2(u(T)) = \alpha(u), \end{cases} \tag{1}$$

where $l_1, l_2 : E \to E'$ are linear bounded maps, $\alpha : AC \multimap E'$ is a continuous map, (E' is a Banach space) then we reformulate it to the following:

$$\begin{cases} y(t) = f(t, z + \int_0^t y(s)ds, y(t)) \\ l_1(z) + l_2(z + \int_0^T y(s)ds) = \alpha(z + \int_0^{\cdot} y(s)ds). \end{cases} \tag{2}$$

Obviously, if $(z, y) \in E \times AC$ is a solution to the problem (2), then $u(t) = z + \int_0^t y(s)ds$ is a solution to the problem (1).

Putting

$$x = (z, y),$$

$$L(z, y) = (y, l_1(z) + l_2(z))$$

and

$$F(z, y) = \left(f(\cdot, z + \int_0^{\cdot} y(s)ds, y(\cdot)), \alpha(z + \int_0^{\cdot} y(s)ds)) - l_2(\int_0^T y(s)ds) \right)$$

we arrive at a coincidence problem (a generalized fixed point problem) of the form

$$L(x) = F(x). \tag{3}$$

Such coincidence problems have been intensively studied by many authors, especially in case when F is a compact (single- or multivalued)map and L is the identity (the Leray-Schauder fixed point theory) or L is a Fredholm operator of index 0 (e.g. Mawhin [9], Pruszko [10]) or of nonnegative index (Kryszewski [8]). The situation when L is a Fredholm operator of nonnegative index and F belongs to a more general class of nonlinear (single- or multivalued) transformations, so called L–fundamentally contractible maps was investigated in [4]. We use some theoretical results from this paper, but not in the most general case (i.e. only for singlevalued maps) .

Observe that in case $E = E'$, $l_1 = id_E$, $l_2 = -id_E$ and $\alpha \equiv 0$, (1) becomes an ordinary periodic boundary value problem.

In Section 1 we introduce some notions and cite a few results and in Section 2 we carefully describe and solve our problem.

Throughout the paper we will use the following notation: if U is a subset of a Banach space E, then by $\mathrm{cl}\, U$ we mean the closure of U, by $\mathrm{bd}\, U$ - the boundary of U, $\mathrm{conv}\,(U)$ - the convex hull of U and $\overline{\mathrm{conv}}\,(U) = \mathrm{cl}\,\mathrm{conv}\,(U)$. Moreover, let $B^E(x_0, r) = \{x \in E; ||x_0 - x||_E \le r\}$ and if $E = \mathbb{R}^n$, then $B^n(x_0, r) := B^{\mathbb{R}^n}(x_0, r)$.

2. Preliminaries

Let E, E' be Banach spaces with norms $|| \cdot ||_E$, $|| \cdot ||_{E'}$, respectively. A bounded linear map $L : E \to E'$ is a *Fredholm operator* if dimensions of its kernel ($\mathrm{Ker}\, L$) and cokernel ($\mathrm{Coker}\, L := E'/\mathrm{Im}\,(L)$, where $\mathrm{Im}\,(L)$ is the image of L), are finite. By the index of a Fredholm operator L we mean the number

$$i(L) := \dim \mathrm{Ker}\, L - \dim \mathrm{Coker}\, L.$$ [1]

Since both $\mathrm{Ker}\,(L)$ and $\mathrm{Im}\,(L)$ are direct summands in E and E', respectively, we may consider continuous linear projections $P : E \to E$ and $Q : E' \to E'$, such

[1] Observe that if $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then L is Fredholm and $i(L) = m - n$.

that $\operatorname{Ker} L = \operatorname{Im}(P)$ and $\operatorname{Ker} Q = \operatorname{Im}(L)$. Clearly E, E' split into (topological) direct sums

$$\operatorname{Ker}(P) \oplus \operatorname{Ker}(L) = E, \qquad \operatorname{Im}(Q) \oplus \operatorname{Im}(L) = E'.$$

Moreover, since $\operatorname{Im}(L)$ is a closed subspace of E', $L|_{\operatorname{Ker} P} : \operatorname{Ker} P \to \operatorname{Im} L$ is a linear homeomorphism onto $\operatorname{Im}(L)$. Denote by K_P the inverse isomorphism for $L|_{\operatorname{Ker} P}$. Note also that L is proper when restricted to a closed bounded set.

Consider a continuous map $F : X \to E'$, where $X \subset E$.

Definition 1. A closed convex and nonempty set $K \subset E'$ is called *L-fundamental* for F, provided

(i) $F(L^{-1}(K) \cap X) \subset K$; and
(ii) if for $x \in X$, $L(x) \in \overline{\operatorname{conv}}(F(x) \cup K)$, then $L(x) \in K$.

It is clear that for any F some L-fundamental set exists (for instance whole E' or $\overline{\operatorname{conv}}(F(X))$).

Observe that if $E = E'$ and $L = id_E$ is the identity on E, then K is nothing else but a fundamental set for F in the sense of e.g. [2] (see also references therein).

Some properties of L-fundamental sets are summarized in the following result (comp. [4] or [5]).

Proposition 1.

(i) If K is an L-fundamental set for F, then $\{x \in X \mid L(x) = F(x)\} \subset L^{-1}(K)$.
(ii) If K_1, K_2 are L-fundamental sets for F, then the set $K := K_1 \cap K_2$ is L-fundamental or empty.
(iii) If $P \subset K$ and K is an L-fundamental set for F, then so is $K' = \overline{\operatorname{conv}}(F(L^{-1}(K) \cap X) \cup P)$.
(iv) If K is the intersection of all L-fundamental sets for F, then

$$K = \overline{\operatorname{conv}}(F(L^{-1}(K) \cap X)).$$

(v) For any $A \subset E'$, there exists an L-fundamental set K such that $K = \overline{\operatorname{conv}}(F(L^{-1}(K) \cap X) \cup A)$.

Definition 2. We say that F is an *L-fundamentally restrictible map* if for any $y \in E'$ there exists a *compact* L-fundamental set for F, which contains y.

Let us collect some important examples of L-fundamentally restrictible maps.

Example 1. Let $L : E \to E'$ be an arbitrary Fredholm operator.

(i) if $F : X \to E'$ is compact (i.e. $\operatorname{cl} F(X)$ is compact), then $K = \overline{\operatorname{conv}}(F(X) \times \{y\})$ is a compact L- fundamental set for F; hence F is L-fundamentally restrictible.

(ii) Let μ be a measure of noncompactness in E' having usual properties (see e.g. [1]) and let F be L-condensing in the sense that, for any bounded set $A \subset X$, if $\mu(F(A)) \geq \mu(L(A))$, then A is compact. If F is bounded, then one shows that an L-fundamental set K, satisfying $K = \overline{\operatorname{conv}}\,(F(L^{-1}(K) \cap X) \cup \{y\})$ for some $y \in E'$ (see Proposition 1) is compact; hence F is L-fundamentally restrictible.

(iii) If F is an L-set contraction (i.e. there exists $k \in (0,1)$, such that for any bounded $A \subset X$, $\mu(F(A)) \leq k\mu(L(A))$), then F is L-condensing and therefore L-fundamentally restrictible.

Some other examples one can find in [4] and in [5].

Now we are going to sketch the construction of a generalized index of coincidence between L and an L-fundamentally restrictible map F. More details (in more general, multivalued case), one can find e.g. in [3] or in [5].

Let U be an open bounded subset of $\mathbb{R}^m$ and let $F : \operatorname{cl} U \to \mathbb{R}^n$, where $m \geq$ $\geq n \geq 1$ and suppose that $0 \notin F(x)$ for $x \in \operatorname{bd} U$. It implies that there is $\varepsilon > 0$ such that $F(\operatorname{bd} U) \subset \mathbb{R}^n \setminus B^n(0, \varepsilon)$.

We can of course define the Brouwer degree for such map, but if $m > n$ it is useless, because always equal to 0. Better homotopy invariant defined Kryszewski (comp. [8]), developing some ideas from [6]. In this definition he used cohomotopy sets. Consider the following sequence of maps:

$$\pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\varepsilon)) \xrightarrow{F^{\#}} \pi^n(\operatorname{cl} U, \operatorname{bd} U) \xleftarrow{i_1^{\#}} \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus U) \xrightarrow{i_2^{\#}}$$
$$\xrightarrow{i_2^{\#}} \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)),$$

where $r > 0$ is such that $U \subset B^m(0,r)$ and $i_1 : (\operatorname{cl} U, \operatorname{bd} U) \to (\mathbb{R}^m, \mathbb{R}^m \setminus U)$ and $i_2 : (\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)) \to (\mathbb{R}^m, \mathbb{R}^m \setminus U)$ are inclusions. Arrows denote maps between cohomotopy sets induced by respective maps (see [7]). By the excision property $i_1^{\#}$ is a bijection. Hence we have defined the transformation

$$\begin{aligned} \mathcal{K} := i_2^{\#} \circ (i_1^{\#})^{-1} \circ F^{\#} : \pi^n(S^n) = \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\varepsilon)) \to \\ \to \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)) = \pi^n(S^m). \end{aligned} \tag{4}$$

Definition 3. By the *generalized degree* of the map F on U we understand the element

$$\deg((F, U, 0) := \mathcal{K}(\mathbf{1}) \in \pi^n(S^m).$$

($\mathbf{1}$ denotes the generator of $\pi^n(S^n) \cong \mathbb{Z}$, i.e. the homotopy class of the identity map $id : S^n \to S^n$.)

It is clear that this definition does not depend on the choice of ε and r.

Remark 1. One can check that if $n = m$, then $\deg(F, U, 0) \in \pi^n(S^n)$ is nothing else but the ordinary Brouwer degree of the map F (comp. the Hopf theorem [7], th.11.5).

Now we are going to define a generalized index of coincidence between a Fredholm operator L of index $i(L) = k$ and an L-fundamentally restrictible map $F : X \to E'$, where X is open subset of E and E, E' are Banach spaces. Suppose that $C := \{x \in X \mid L(x) \in F(x)\}$ is bounded and closed. Therefore there is an open bounded set U such that $C \subset U \subset \operatorname{cl} U \subset X$. Let K_0 be *any* compact L-fundamental set for F. In view of Proposition 1 (i), C is contained in $L^{-1}(K_0) \cap \operatorname{cl} U$. Since $L|_{\operatorname{cl} U}$ is proper, we gather that C being obviously closed is also compact. Now let consider a map

$$F_{|(L^{-1}(K_0)\cap X)} : L^{-1}(K_0) \cap X \to E',$$

According to Definition 1, the range of this map is contained in K_0. Hence it has a compact extension

$$\overline{F} : X \to K_0{}^2.$$

It is clear that $\{x \in X \mid L(x) = \overline{F}(x)\} = C$.

There is $\varepsilon_0 > 0$ such that

$$\{y \in E' \mid \exists_{x \in \operatorname{bd} U} \quad y = L(x) - F(x)\} \cap B^{E'}(0, 2\varepsilon_0) = \emptyset.$$

Take $\varepsilon \in (0, \varepsilon_0]$ and let $l_\varepsilon : \operatorname{cl} \overline{F}(U) \to E'$ be a Schauder projection of the compact set $\operatorname{cl} \overline{F}(U)$ into a finite dimensional subspace Z of E', such that $||l_\varepsilon(y) - y||_{E'} < \varepsilon$ for $y \in \operatorname{cl} \overline{F}(U)$. Denote by W' the finite dimensional subspace of $\operatorname{Im}(L)$ such that $Z \subset W = W' \oplus \operatorname{Im}(Q)$. Put $T := L^{-1}(W)$, $U_W = U \cap T$. It is clear that the closure $\operatorname{cl} U_W$ (in T) is contained in $\operatorname{cl} U \cap T$ and its boundary $\operatorname{bd} U_W$ (relative T) in $\operatorname{bd} U \cap T$. Further let $\overline{F_W} = l_\varepsilon \circ \overline{F}|_{\operatorname{cl} U_W}$ and $L_W = L|_T : T \to W$. Observe, that L_W is a Fredholm operator of index

$$i(L_W) = \dim T - \dim W = k = i(L).$$

Enlarging W' if necessary we may assume that $\dim W := n \geq k + 2$. Putting $m := \dim T = n + k$ we arrive in a finite dimensional situation discussed above.

Definition 4. By the *generalized index of the L-fundamentally restrictible map* F we understand the element

$$\operatorname{Ind}_L(F, X) := \deg(L_W - \overline{F_W}), U_W, 0) \in \Pi_k.$$

By definition, $\deg(L_W - \overline{F_W}, U_W, 0)$ belongs to $\pi^n(S^m)$ but since $m < 2n - 1$ we know that $\pi^n(S^m) \cong \Pi_k$.

One can check (see [5] or [3]) that the definition does not depend on the choice of a compact L-fundamental set K_0, an extension $\overline{F}$ of $F|_{L^{-1}(K_0)\cap X}$, an open subset U, a number $\varepsilon \in (0, \varepsilon_0]$, a projection l_ε and a space W'.

[2] For instance one can take any retraction $r : E' \to K_0$ and define $\overline{F} := r \circ F$.

Definition 5. Given L-fundamentally restrictible maps F_0, F_1 we say that they are (L,K)*–homotopic* (written $F_0 \simeq_K F_1$) if there is a homotopy $H : X \times [0,1] \to E'$ such that the set $\{x \in X \mid L(x) \in H(x,t))$ for some $t \in [0,1]\}$ is bounded and closed in E and K is a compact L-fundamental set for any map $X \ni x \mapsto H(x,t)$ where $t \in [0,1]$.

At the first glance the above definition of homotopic pairs is enough for our next considerations (comp. Theorem 1), but in applications we need the following more general one.

Definition 6. Two L-fundamentally restrictible maps F_0, F_1 are L*-homotopic* if there is a finite number of compact convex sets $K_1, \ldots, K_n$ and L-fundamentally restrictible maps $G_1, \ldots, G_{n-1}$ such that

$$F_0 \simeq_{K_1} G_1 \simeq_{K_2} \cdots \simeq_{K_{n-1}} G_{n-1} \simeq_{K_n} F_1.$$

Theorem 1. *The generalized index* Ind_L *on has the following properties (as above,* $C := \{x \in X \mid L(x) \in F(x)\}$*):*

(i) (Existence) *If* $\mathrm{Ind}_L(F,X) \neq 0$*, then there is* $x \in X$ *such that* $L(x) \in F(x)$*.*
(ii) (Localization) *If* $X' \subset X$ *is open and* $C \subset X'$*, then* $\mathrm{Ind}_L(F,X')$ *is defined and equal to* $\mathrm{Ind}_L(F,X)$*.*
(iii) (Homotopy Invariance) *If* F_0*,* F_1 *are* L*-homotopic, then* $\mathrm{Ind}_L(F_0,X) = \mathrm{Ind}_L(F_1,X)$*.*
(iv) (Additivity) *If* X_1*,* X_2 *are open disjoint subsets of* X *such that* $C \subset X_1 \cup X_2$*, then*

$$\mathrm{Ind}_L(F,X) = \mathrm{Ind}_L((F,X_1) + \mathrm{Ind}_L(F,X_2).$$

(v) (Restriction) *If* $F(X)) \subset Y$*, where* $Y \subset Y' \oplus \mathrm{Im}\,(Q)$ *is a closed subspace of* E'*, then* $\mathrm{Ind}_L(F,X) = \mathrm{Ind}_{L_Y}(F_Y, X \cap T)$*, where* $T := L^{-1}(Y' \oplus \mathrm{Im}\,(Q))$*,* $F_Y = F|_{X\cap T)}$ *and* $L_Y = L|_T$*.*

The proof can be found in [4] or in [3].

Applying the coincidence index constructed above, we present in the following theorem conditions sufficient for the existence of solutions to the abstract coincidence problem

$$L(x) = F(x), \tag{5}$$

where $L : E \to E'$ is a Fredholm linear operator of nonnegative index k (E, E' are Banach spaces) and F is a continuous map. This result is a slight modification of Theorem 4.1 in [4] (see Remark 4.2 therein), where the proof is included. Let P and Q be respective projections defined for L, I' be the identity map on E' and let $\mathrm{Im}\,Q \neq \{0\}$.

Theorem 2. *Let* $F : E \multimap E'$ *be a map such that*

(i) *there exists an open bounded subset* V *of* E *such that, for any* $x \in E \setminus V$ *and* $\lambda \in [0,1]$, $0 \notin ((1-\lambda)(I' - Q) + Q) \circ F(x)$, *and* $F|_{\mathrm{cl}\, V}$ *is an* L*-fundamentally restrictible map with some* L*-fundamental set containing* 0,

(ii) $\mathrm{Ind}_{\mathcal{O}}(Q \circ F|_{V \cap \mathrm{Im}\, P}, V \cap \mathrm{Im}\, P)$ *is nontrivial* ($\mathcal{O} : \mathrm{Im}\,(P) \to \mathrm{Im}\,(Q)$ *is a Fredholm operator such that* $\mathcal{O}(v) = 0$ *for all* $v \in \mathrm{Im}\,(P)$).

Then the problem $L(x) = F(x)$ *has a solution.*

3. Boundary value problem

Below we illustrate the above result by the boundary value problem.

Let E, E' be Banach spaces with Hausdorff measures of noncompactness [3] χ and χ' respectively and Z be the set of all positive numbers k such that the Fredholm linear operator $D : E \to E'$ is (k, χ, χ')-set contraction[4]. Following [1] we define

$$||D||^{(\chi,\chi')} := \inf Z.$$

Note that $||D||^{(\chi,\chi')} \leq ||D||$.

Denote $J = [0,T] \subset \mathbb{R}$ and let ξ be a Hausdorff measure of noncompactness in the space $\mathcal{L} = L^1(J,E)$ of integrable functions in the sense of Bochner with the norm $\|u\|_{\mathcal{L}} = \int_0^T \|u(s)\|_E \, ds$.

Let $f : J \times E \times E \to E$ be a map satisfying the following assumptions:

(f_1) $f(\cdot, u, v)$ is a measurable map for every $(u,v) \in E \times E$, and $f(t,\cdot,\cdot)$ is continuous for almost all $t \in J$,

(f_2) there are two continuous functions $\lambda_1, \lambda_2 : J \to [0,\infty)$ such that, for any $u_1, u_2, v_1, v_2 \in E$ and almost all $t \in J$,

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\|_E \leq \lambda_1(t)\|u_1 - u_2\|_E + \lambda_2(t)\|v_1 - v_2\|_E,$$

(f_3) there are integrable functions $m, n : J \to [0,\infty)$ such that $\|f(t,u,v)\|_E \leq m(t) + n(t)\|u\|_E$ for any $u, v \in E$ and almost all $t \in J$.

Let us consider the following boundary value problem

$$\begin{cases} u'(t) = f(t, u(t), u'(t)) & \text{for a.a. } t \in J, \\ A_1(u(0)) + A_2(u(T)) = \alpha(u(0)), \end{cases} \tag{6}$$

where f satisfies assumptions (f_1)-(f_3), α is a continuous compact map, and $A_1, A_2 : E \to E'$ are linear operators such that $A := A_1 + A_2$ is a Fredholm operator of nonnegative index. By a solution of problem (6) we mean an absolutely continuous map satisfying the equation for a.a. $t \in J$ an the boundary condition.

[3] Recall that χ is a Hausdorff measure of noncompactness on a space Banach E if for any bounded set $A \subset E$, $\chi(A) = \inf\{\varepsilon \mid A \text{ has a finite } \varepsilon\text{-net}\}$

[4] i.e. for any bounded set $B \in E$, the set $D(B)$ is bounded and $\chi'(D(B)) \leq k\chi(B)$.

The problem (6) is equivalent to the following one:

$$L(z,y) = F(z,y), \tag{7}$$

where $L, F : E \times \mathcal{L} \to E' \times \mathcal{L}$ and

$$L(z,y) = (A(z), y)$$

$$F(z,y) = \left(\alpha(z) - A_2\Big(\int_0^T y(s)ds\Big), f\Big(\cdot, z + \int_0^{(\cdot)} y(s)ds,\ \ y(\cdot)\Big)\right).$$

In fact, (z,y) is a solution of the coincidence problem (7) iff the map $u \in \mathcal{L}$, $u(t) := z + \int_0^t y(s)ds$ is a solution of (6).

Assume that in the spaces $E \times \mathcal{L}$ and $E' \times \mathcal{L}$ we have the norms $\|(z,y)\|_1 = \max(\|z\|_E, \|y\|_{\mathcal{L}})$ and $\|(z',y)\|_2 = \max(\|z'\|_{E'}, \|y\|_{\mathcal{L}})$, respectively. Denote by μ and μ' the Hausdorff measures of noncompactness in $E\times\mathcal{L}$ and $E'\times\mathcal{L}$, respectively, and by pr_E and $pr_{\mathcal{L}}$ (resp. $pr_{E'}$ and $pr'_{\mathcal{L}}$) projections of the space $E \times \mathcal{L}$ (resp. $E' \times \mathcal{L}$) onto E and onto $\mathcal{L}$ (resp. onto E' and $\mathcal{L}$). Observe that if S is a bounded subset of $E \times \mathcal{L}$, then $\mu(S) = \max(\chi(pr_E(S)), \xi(pr_{\mathcal{L}}(S)))$.

Let $N = \int_0^T n(s)ds$, $M = \int_0^T m(t)dt$, $\Lambda_1 = \int_0^T \lambda_1(s)ds$, $\Lambda_2 = \sup_{t\in J}\lambda_2$ and let P_A, Q_A i K_{P_A} be the respective projections and the right inverse for A.

Theorem 3. *Assume that f satisfies assumptions $(f_1)-(f_3)$, the maps α and A are as above, and $Q_A \not\equiv 0$. Moreover, let*
(f_4) $\Lambda_2 < 1$ and $\Lambda_1(1 + \|K_{P_A}\|^{(\chi',\chi)}) < 1 - \Lambda_2$,
(f_5) $\|A_2\| < 1$,
(f_6) $\|K_{P_A}\| \cdot N\exp(N) < 1$,
(f_7) $\operatorname{Im} A_2 \subset \operatorname{Im} A$,
(f_8) there exists $R > 0$ such that, for every $z \in E$ satisfying $\|P_A(z)\|_E \geq R$, $Q_A(\alpha(z)) \neq 0$ and $\operatorname{Ind}_{\mathcal{O}}(Q_A\circ\alpha, B^E(0,R)\cap \operatorname{Im} P_A) \neq 0$, where $\mathcal{O} : \operatorname{Im} P_A \to \operatorname{Im} Q_A$ and $\mathcal{O} \equiv 0$

Then problem (7) has a solution.

Assumptions (f_4) and (f_5) will secure that F is L-condensing, while (f_6)–(f_8) will allow us to check that a generalized index of F is nontrivial, which will imply the existence of a solution to problem (7).

Proof. We show that L and F satisfy assumptions of Theorem 1. For clarity we divide the proof into some steps but first of all, notice that L is a Fredholm operator of index $i(L) = i(A) \geq 0$. Respective projections and the right inverse of L will be denoted in a standard way by P, Q and K_P. The following equalities hold: $\operatorname{Ker} L = \operatorname{Ker} A \times \{0\}$, $\operatorname{Ker} P = \operatorname{Ker} P_A \times \mathcal{L}$, $\operatorname{Im} L = \operatorname{Im} A \times \mathcal{L}$ and $\operatorname{Im} Q = \operatorname{Im} Q_A \times \{0\}$.

STEP 1. We prove that F is continuous.

Let $(z_0, y_0) \in E \times \mathcal{L}$ and $\varepsilon > 0$ be arbitrary. By the continuity of α, there is $\delta_1 > 0$ such that $\|\alpha(z_0) - \alpha(z)\|_{E'} < \frac{\varepsilon}{4}$ for $\|z_0 - z\|_E < \delta_1$.

Take

$$\delta < \min(\delta_1, \frac{\varepsilon}{4\|A_2\|}, \frac{\varepsilon}{8\Lambda_1}, \frac{\varepsilon}{4\Lambda_2}) \tag{8}$$

and assume that for some $(z, y) \in E \times \mathcal{L}$,

$$\begin{aligned}\delta > \|(z_0, y_0) - (z, y)\|_{E\times\mathcal{L}} = \max(\|z_0 - z\|_E, \|y_0 - y\|_{\mathcal{L}}) = \\ = \max\left(\|z_0 - z\|_E, \int_0^T \|y_0(s) - y(s)\|_E ds\right).\end{aligned}$$

Since

$$\begin{aligned}\|F(z_0, y_0) - F(z, y)\|_{E'\times\mathcal{L}} = \\ \max\left(\left\|\alpha(z_0) - A_2\left(\int_0^T y_0(s)ds\right) - \alpha(z) + A_2\left(\int_0^T y(s)ds\right)\right\|_{E'},\right. \\ \left.\left\|f\left(\cdot, z_0 + \int_0^{\cdot} y_0(s)ds, y_0(\cdot)\right) - f\left(\cdot, z + \int_0^{\cdot} y(s)ds, y(\cdot)\right)\right\|_{\mathcal{L}}\right)\end{aligned}$$

and one can check, from (8), that

$$\left\|\alpha(z_0) - A_2\left(\int_0^T y_0(s)ds\right) - \alpha(z) + A_2\left(\int_0^T y(s)ds\right)\right\|_{E'} \leq \frac{\varepsilon}{2},$$

$$\left\|f(\cdot, z_0 + \int_0^{\cdot} y_0(s)ds, y_0(\cdot)) - f(\cdot, z + \int_0^{\cdot} y(s)ds, y(\cdot))\right\|_{\mathcal{L}} \leq \frac{\varepsilon}{2},$$

we obtain

$$\|F(z_0, y_0) - F(z, y)\|_{E'\times\mathcal{L}} < \varepsilon,$$

which implies a continuity of F.

STEP 2. We show that for any open bounded subset V of $E \times \mathcal{L}$, the set $F(V)$ is also bounded, and $F|_{\mathrm{cl}\,V}$ is L-condensing (so, L-fundamentally restrictible).

Let S be an arbitrary subset of V. We check that $\mu'(F(S)) < \mu'(L(S))$. Let $\chi(pr_E(S)) = \varepsilon$ and $\xi(pr_{\mathcal{L}}(S)) = \delta$. Then

$$\mu'(L(S)) = \max\left[\chi'\big(pr_{E'}(L(S))\big), \xi\big(pr_{\mathcal{L}}(L(S))\big)\right] = \max\left[\chi'\big(pr_{E'}L(S))\big), \delta\right].$$

Since $\operatorname{Ker} L = \operatorname{Im} P_A$ is a finite dimensional space,

$$\chi(pr_E(S)) = \chi((I_E - P_A) \circ pr_E(S)) = \chi(K_{P_A} \circ A \circ pr_E(S)).$$

One knows that

$$\chi(K_{P_A} \circ A \circ pr_E(S)) \leq \|K_{P_A}\|^{(\chi',\chi)} \chi'(A(pr_E(S)))$$

and

$$pr_{E'}(L(S)) = A(pr_E(S)),$$

thus

$$\chi'(pr_{E'}(L(S))) \geq \frac{\chi(pr_E(S))}{\|K_{P_A}\|^{(\chi',\chi)}} = \frac{\varepsilon}{\|K_{P_A}\|^{(\chi',\chi)}}.$$

This implies

$$\mu'(L(S)) \geq \max\left[\frac{\varepsilon}{\|K_{P_A}\|^{(\chi',\chi)}}, \delta\right].$$

Now, calculate $\mu'(F(S))$. Obviously,

$$\mu'(F(S)) = \max\Bigg(\chi'\Big(\Big\{\alpha(z) - A_2\Big(\int_0^T y(s)ds\Big); (z,y) \in S\Big\}\Big), \xi\Big(\Big\{f\Big(\cdot, z + \int_0^{(\cdot)} y(s)ds, y(\cdot)\Big); (z,y) \in S\Big\}\Big)\Bigg).$$

Since α is a compact map, $\chi'(\{\alpha(z) | z \in pr_E(S)\}) = 0$, hence, by a suitable property of measures of noncompactness,

$$\chi'\Big(\Big\{\alpha(z) - A_2\Big(\int_0^T y(s)ds\Big) | (z,y) \in S\Big\}\Big) \leq \chi'\Big(\Big\{A_2\Big(\int_0^T y(s)ds\Big) | y \in \mathrm{pr}_{\mathcal{L}}(S)\Big\}\Big).$$

For every $\delta_1 > 0$ there is a finite $(\delta + \delta_1)$–net in $pr_{\mathcal{L}}(S)$. Let y_k be an arbitrary element of this net. If $\|y_k - y\|_{\mathcal{L}} \leq \delta + \delta_1$ for some $y \in pr_{\mathcal{L}}(S)$, then

$$\begin{aligned}\left\|A_2\left(\int_0^T y_k(s)ds\right) - A_2\left(\int_0^T y(s)ds\right)\right\|_{E'} &= \left\|A_2\left(\int_0^T y_k(s) - y(s)ds\right)\right\|_{E'} \leq \\ &\leq \|A_2\| \cdot \left\|\int_0^T (y_k(s) - y(s))ds\right\|_E \leq \|A_2\| \cdot \int_0^T \|y_k(s) - y(s)\|_E ds = \\ &= \|A_2\| \cdot \|y_k - y\|_{\mathcal{L}} < \\ &< \|A_2\|(\delta + \delta_1).\end{aligned}$$

Therefore $\chi'(A_2(\{\int_0^T y(s)ds | y \in pr_{\mathcal{L}}(S)\})) \leq \|A_2\|\delta < \delta$, what implies that

$$\mu'(F(S)) \leq \max\Bigg(\delta, \xi\Big(\Big\{f\Big(\cdot, z + \int_0^{(\cdot)} y(s)ds, y(\cdot)\Big); (z,y) \in S\Big\}\Big)\Bigg).$$

Analogously, for every $\varepsilon_1 > 0$ there is a finite $(\varepsilon+\varepsilon_1)$–net in $pr_E(S)$. Let z_l be its arbitrary element. If $\|y_k - y\|_{\mathcal{L}} \le \delta + \delta_1$ and $\|z_l - z\| \le \varepsilon + \varepsilon_1$ hold for some $y \in pr_{\mathcal{L}}(S)$ and $z \in pr_E(S)$, then

$$\int_0^T \left\| f\left(t, z_l + \int_0^t y_k(s)ds, y_k(t)\right) - f\left(t, z + \int_0^t y(s)ds, y(t)\right)\right\|_E dt \le$$
$$\le \int_0^T \left(\lambda_1(t)\left\| z_l + \int_0^t y_k(s)ds - z - \int_0^t y(s)ds\right\|_E + \lambda_2(t)\|y_k(t)-y(t)\|_E\right) dt \le$$
$$\int_0^T\!\!\left(\lambda_1(t)\left(\|z_l - z\|_E + \| \int_0^t y_k(s)ds - \int_0^t y(s)ds\|_E\right) + \lambda_2(t)\|y_k(t)-y(t)\|_E\right) dt \le$$
$$\le \int_0^T \lambda_1(t)(\varepsilon+\varepsilon_1+\delta+\delta_1)dt + \int_0^T \lambda_2(t)\|y_k(t)-y(t)\|_E dt \le$$
$$\le \varLambda_1(\varepsilon+\varepsilon_1+\delta+\delta_1) + \varLambda_2(\delta+\delta_1).$$

Since ε_1 and δ_1 was arbitrary, we have

$$\xi(pr'_{\mathcal{L}}(F(S))) \le \varLambda_1(\varepsilon+\delta) + \varLambda_2\delta,$$

and consequently, using (f_4),

$$\mu'(F(S)) = \max\left(\chi'\left(pr_{E'}(F(S))\right), \xi\left(pr'_{\mathcal{L}}(F(S))\right)\right) < \max\left(\delta, \frac{\varepsilon}{\|K_{P_A}\|^{(\chi',\chi)}}\right) \le$$
$$\le \mu'(L(S)).$$

This implies that $F|_{\mathrm{cl}\,V}$ is L-condensing map, hence there exists a compact L-fundamental set for $F|_{\mathrm{cl}\,V}$ containing 0.

STEP 3. We prove that, for some open bounded set $V \subset E \times \mathcal{L}$, the map $((1-\lambda)(I-Q)+Q)\circ F$ has no coincidence points with L outside V (I denotes the identity map in $E' \times \mathcal{L}$). Let I_E, $I_{E'}$ be the identity maps on spaces E, E' respectively

Let $Z > 0$ be such that $\alpha(E) \subset B^E(0,Z)$. Choose $R_1 > 0$ such that

$$R_1 > \frac{\|K_{P_A}\|(Z + M\exp(N) + NR\exp(N))}{1 - \|K_{P_A}\|N\exp(N)}$$

and let

$$R_2 := (M + N(R+R_1))\exp(N).$$

Define

$$V{:=}\Big\{(z,y)\in E\times\mathcal{L} \mid \; P_A(z)\in B^E(0,R)\cap \operatorname{Ker} A,$$
$$(I_E - P_A)(z)\in B^E(0,R_1)\cap \operatorname{Ker} P_A, \;\; y\in B^{\mathcal{L}}(0,R_2)\Big\}.$$

Suppose, on the contrary, that there is $\lambda \in [0,1]$ such that

$$L(z,y) = ((1-\lambda)(I-Q)+Q)\circ F(z,y).$$

It follows that $Q \circ F(z,y) = 0$, since $L(z,y) \in (I-Q)(E' \times \mathcal{L})$. Moreover,

$$((1-\lambda)(I-Q)+Q)\circ F(z,y) = \\ = \left(\Big((1-\lambda)(I_{E'}-Q_A)+Q_A\Big)\left(\alpha(z) - A_2(\int_0^T y(s)ds)\right), f(\cdot, z+\int_0^{\cdot} y(s)ds, y(\cdot))\right),$$

so we obtain that:

$$y(\cdot) = f\left(\cdot, z + \int_0^{\cdot} y(s)ds, y(\cdot)\right), \tag{9}$$

$$A(z) = (1-\lambda)(I_{E'}-Q_A)\left(\alpha(z) - A_2(\int_0^T y(s)ds)\right), \tag{10}$$

and

$$Q_A\left(\alpha(z) - A_2(\int_0^T y(s)ds)\right) = 0. \tag{11}$$

The last equality and assumption (f_7) imply $Q_A(\alpha(z)) = 0$, so by (f_8),

$$\|P_A(z)\|_E < R. \tag{12}$$

Consider the continuous map $[0,T] \ni t \mapsto \int_0^t \|y(s)\|ds$. From equality (9) and assumption (f_3) it follows that

$$\int_0^t \|y(s)\|_E ds = \int_0^t \left\| f\left(s, z + \int_0^s y(\tau)d\tau, y(s)\right)\right\|_E ds \leq \\ \leq \int_0^t \left(m(s) + n(s)\left\|z + \int_0^s y(\tau)d\tau\right\|_E\right) ds \leq \int_0^t m(s)ds + \\ + \int_0^t n(s)(\|P_A(z)\|_E + \|(I_E - P_A)(z)\|_E)ds + \int_0^t (n(s)\int_0^s \|y(r)\|_E dr)ds,$$

and, by the Gronwall inequality,

$$\int_0^t \|y(s)\|_E ds \leq (M + N(\|P_A(z)\|_E + \|(I_E - P_A)(z)\|_E))\exp(\int_0^t n(s)ds) \leq \\ \leq (M + N(\|P_A(z)\|_E + \|(I_E - P_A)(z)\|_E))\exp(N).$$

Combining this with (12) one obtains

$$\|y\|_{\mathcal{L}} \leq (M + N(R + \|(I_E - P_A)(z)\|_E)) \exp(N). \tag{13}$$

Since $(I_E - P_A)(z) = K_{P_A} \circ A(z)$, conditions (10), (13) and assumption (f_5) imply that

$$\|(I_E - P_A)(z)\|_E = (1-\lambda) \left\| K_{P_A} \circ (I_{E'} - Q_A) \left(\alpha(z) - A_2(\int_0^T y(s)ds) \right) \right\|_E \leq$$

$$\leq (1-\lambda)\left(\|K_{P_A} \circ (I_{E'} - Q_A)(\alpha(z))\|_E + \|K_{P_A} \circ (I_{E'} - Q_A) \circ A_2(\int_0^T y(s)ds)\|_E \right) \leq$$

$$\leq (1-\lambda)\left(\|K_{P_A}\| \cdot \|(I_{E'} - Q_A)(\alpha(z))\|_{E'} + \right.$$

$$\left. + \|K_{P_A}\| \cdot \|I_{E'} - Q_A\| \cdot \|A_2\| \cdot \| \int_0^T y(s)ds\|_E \right) \leq$$

$$\leq ((1-\lambda)\|K_{P_A}\| \cdot (Z + (M + N(R + \|(I_E - P_A)(z)\|_E)) \exp(N)) .$$

Now, if $\lambda = 1$, then $\|(I_E - P_A)(z)\| = 0 < R_1$ and if $0 \leq \lambda < 1$, then also (using the above inequalities)

$$\|(I_E - P_A)(z)\|_E \leq \frac{\|K_{P_A}\|(Z + M\exp(N) + NR\exp(N))}{1 - \|K_{P_A}\| N \exp(N)} \leq R_1, \tag{14}$$

which jointly with (13) implies

$$\|y\|_{\mathcal{L}} < (M + N(R + R_1)) \exp(N) = R_2. \tag{15}$$

By inequalities (12), (14) and (15) we can conclude that all coincidence points of L and maps $((1-\lambda)(I - Q) + Q) \circ F$, where $\lambda \in [0, 1]$, are contained in V.

STEP 4. We use assumptions (f_7) and (f_8) to obtain that, for every $(z, y) \in V$,

$$Q \circ F|_{V \cap \operatorname{Im} P}(z, y) = Q(\alpha(z), 0) = Q_A(\alpha(z)),$$

and hence, $\operatorname{Ind}_{\mathcal{O}}(Q \circ F|_{V \cap \operatorname{Im} P}, V \cap \operatorname{Im} P)$ is nontrivial.

Resuming, in succeeding steps we have proved that the Fredholm operator L and the map F satisfy the assumptions of Theorem 1, so problem (6) has a solution.

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