ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 329–342, CDDE 2000 issue

OSCILLATION THEORY OF LINEAR DIFFERENCE EQUATIONS

Ondřej Došlý

Mathematical Institute, Academy of Sciences of the Czech Republic
Žižkova 22, CZ-616 62 Brno, Czech Republic
Email: dosly@math.muni.cz

Abstract. The survey of the basic results of oscillation theory of various linear differential equations and systems is presented. It is shown that the discrete oscillation theory is in many aspects very similar to its continuous counterpart. Some open problem are discussed.

AMS Subject Classification. 39A10

Keywords. Discrete oscillation theory, Sturm-Liouville difference equation, Riccati equation, variational principle, symplectic system.

1. Introduction

The aim of this paper is to present a brief survey of the basic results of the discrete oscillation theory, to compare these results with their continuous counterparts, and to formulate some open problems in this area.

Let us start, as a motivation for our investigation, with the very famous second order linear difference equation, namely the equation

$$x_{k+2} = x_{k+1} + x_k \tag{1}$$

which determines the Fibonacci numbers. The characteristic equation of (1) is $\lambda^2 - \lambda - 1 = 0$, hence

$$x_k^{[1]} = \left(\frac{1+\sqrt{5}}{2}\right)^k, \quad x_k^{[2]} = \left(\frac{1-\sqrt{5}}{2}\right)^k$$

is a pair of linearly independent solutions of (1). Obviously, the solution $x^{[1]}$ is a monotonically increasing sequence, whereas $x^{[2]}$ is an oscillatory sequence. From this point of view, it seems that the Sturmian separation theorem concerning the zero points of the linearly independent solutions of the Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0, \quad r(t) > 0, \tag{2}$$

has no discrete analogue.

To show that this is not the case, let us return to the motivation for the investigation of oscillatory properties of (2) (more precisely, distribution of zero points of its solutions). One of these motivations consists in the investigation of positivity of the quadratic functional

$$\mathcal{F}_c(y) := \int_a^b \left[r(t)y'^2 - c(t)y^2\right] dt \tag{3}$$

over the class of (nontrivial, sufficiently smooth) functions y satisfying $y(a) = 0 = y(b)$. The functional $\mathcal{F}$ is (upon a certain transformation) the functional of the second variation of the fixed end points variational problem

$$\int_a^b f(t, x(t), x'(t))\, dt \to \min, \quad x(a) = A, x(b) = B. \tag{4}$$

and its positivity is a sufficient condition for an extremal to be a local minimum of (4), for a more detailed treatment of this topic see [19].

The important role in the investigation of positivity of the functional $\mathcal{F}_c$ is played by the so-called Picone identity. This identity relates the quadratic functional $\mathcal{F}_c$ to the Riccati equation

$$w' + c(t) + \frac{w^2}{r(t)} = 0 \tag{5}$$

which is related to (2) by the substitution $w := \frac{r(t)x'}{x}$. This identity reads as follows; let w be a solution of (5) which exists on the whole interval $[a, b]$, then

$$\mathcal{F}_c(y) = w(t)\, y^2\big|_a^b + \int_a^b \frac{1}{r(t)} \left(r(t)y' - w(t)y\right)^2 dt, \tag{6}$$

in particular, if $y(a) = 0 = y(b)$, this formula shows that the existence of a solution x of (2) without zero in $[a, b]$ (and hence the existence of w solving (5) on $[a, b]$) implies that $\mathcal{F}_c$ can be "completed to the square" (compare the integral term on the right-hand-side of (6)) and hence $\mathcal{F}_c$ is positive over the class of y satisfying $y(a) = 0 = y(b)$.

If we replace the integral in (4) by its partial Riemann sum, after some relabeling of variables in this extremal problem, its discrete version is

$$\sum_{k=0}^{N} f(k, x_{k+1}, \Delta x_k) \to \min, \quad x_0 = A,\ x_{N+1} = B, \tag{7}$$

for a more detailed description of this discretization process we refer to [2,22]. The investigation of sufficient conditions for a local minimum of (7) leads (using essentially the same arguments as in the continuous case) to the problem of positivity of the discrete quadratic functional

$$\mathcal{F}_d(y) := \sum_{k=0}^{N} \left[r_k (\Delta y_k)^2 - c_k y_{k+1}^2 \right], \quad \Delta y_k := y_{k+1} - y_k, \tag{8}$$

in the class of nontrivial sequences $y = \{y_k\}_{k=0}^{N+1}$ satisfying $y_0 = 0 = y_{N+1}$. This functional is connected with the Sturm-Liouville difference equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \quad r_k \neq 0, \tag{9}$$

in the same way as (3) and (2) in the continuous case. The discrete analogue of (5) is the equation

$$\Delta w_k + c_k + \frac{w_k^2}{r_k + w_k} = 0 \tag{10}$$

and this equation is related to (9) by the substitution $w_k = \frac{r_k \Delta x_k}{x_k}$. Here one can see already a certain difference between the discrete and continuous case, namely the presence of w in the denominator of the last expression of (10), we will return to this phenomenon later in this paper. Following the same idea as in the continuous case we reveal the discrete Picone identity

$$\mathcal{F}_d(y) = w_k y_k^2 \big|_0^{N+1} + \sum_{k=0}^{N} \frac{1}{r_k + w_k} \left(r_k \Delta y_k - w_k y_k \right)^2, \tag{11}$$

where w is a solution of (10) defined for every $k \in [0, N+1]$. In particular, the term $r+w$ plays the same role as the term r in the continuous case and hence $\mathcal{F}_d$ is positive (for nontrivial y satisfying $y_0 = 0 = y_{N+1}$) provided there exists a solution w of (10) defined for $k \in [0, N+1]$ and satisfying $w_k + r_k > 0$ for $k \in [0, N]$. Substituting for $w = \frac{r \Delta x}{x}$, the last inequality is equivalent to $r_k x_k x_{k+1} > 0$. Consequently, this leads to the following definition.

Definition 1. We say that an interval $(m, m+1]$, $m \in \mathbb{Z}$, contains a *generalized zero* of a solution x of (9) if $x_m \neq 0$ and $x_m x_{m+1} r_m \leq 0$.

The Fibonacci equation (1) can be rewritten into the (self-adjoint) form

$$\Delta \left((-1)^k \Delta x_k \right) + (-1)^k x_{k+1} = 0,$$

see [2, Chap. I]. Applying the above definition (with $r_k = (-1)^k$) to this equation we easily see that both solutions $x^{[1]}, x^{[2]}$ are actually oscillatory, they have infinitely many generalized zeros.

Finally note that the discrepancies between discrete and continuous oscillation theories are mostly caused by differences between continuous calculus (differential and integral calculus) and its discrete counterpart (the calculus of differences and sums).

2. Oscillation theory of Sturm-Liouville difference equations

Using the definition of a generalized zero from the previous section we can now formulate the main statement of the oscillation theory of Sturm-Liouville difference equations (9), the so-called Roundabout theorem, see e.g. [2].

Theorem 1. *The following statements are equivalent:*

(i) Equation (9) *is disconjugate on* $[0,N]$*, i.e., the solution* $\tilde{x}$ *given by the initial condition* $\tilde{x}_0=0$*,* $r_0\tilde{x}_1=1$ *has no generalized zero in* $(0,N+1]$*.*
(ii) There exists a solution of (9) *having no generalized zero in* $[0,N+1]$*.*
(iii) There exists a solution w *of* (10) *which is defined for every* $k\in[0,N+1]$ *and satisfies* $r_k+w_k>0$ *for* $k\in[0,N]$*.*
(iv) The quadratic functional $\mathcal{F}_d(y)$ *is positive for every nontrivial* y *satisfying* $y_0=0=y_{N+1}$*.*

This theorems shows that the Sturmian separation and comparison theory does extend to (9). Indeed, the separation theorem is given by the equivalence (i) $\iff$ (ii) and the comparison theorem is "hidden" in the equivalence (i) $\iff$ (iv). Let us also remind the main ideas used in the proof of Theorem 1. The implication (i) $\implies$ (ii) follows from the continuous dependence of solutions of (9) on a parameter. More precisely, if the solution $\tilde{x}$ given in (i) has no generalized zero in $(0,N+1]$, then the solution $x^{[\varepsilon]}$ given by the initial condition $x_0^{[\varepsilon]}=\epsilon$, $r_0x_1^{[\varepsilon]}=1$ has no generalized zero in $[0,N+1]$ if $\varepsilon>0$ is sufficiently small. The implication (ii) $\implies$ (iii) is just the Riccati substitution and the already mentioned fact that $r_k+w_k>0$ if and only if $r_kx_kx_{k+1}>0$. The implication (iii) $\implies$ (iv) follows immediately from Picone's identity. Finally, the implication (iv) $\implies$ (i) is proved by contradiction. If $\tilde{x}$ would have a generalized zero in $(0,N+1]$, one can construct a nontrivial $y=\{y_k\}_{k=0}^{N+1}$ with $y_0=0=y_{N+1}$ such that $\mathcal{F}_d(y)\le 0$. More details concerning this proof can be found e.g. in [5].

The Roundabout theorem (observe that this name for the theorem comes from its proof) immediately suggests two main methods of the discrete oscillation theory. The first one consists in the equivalence (i) $\iff$ (iv) and is called the *variational method*, whereas the second method, leaned on the equivalence (i) $\iff$ (iii), is usually referred as the *Riccati technique*. Recall that equation (9) is said to be *nonoscillatory* if there exists $N\in\mathbb{N}$ such that (9) is disconjugate on $[N,M]$ for every $M>N$, in the opposite case (9) is said to be *oscillatory*.

To prove (via the variational method) that (9) is oscillatory, it suffices to construct for every $N\in\mathbb{N}$ a sequence $y=\{y_k\}_{k=N}^{\infty}$, such that $y_N=0$, only finitely many y_k are nonzero (this class of sequence we will denote by $\mathcal{D}(N)$) and

$$\mathcal{F}_d(y;N,\infty):=\sum_{k=N}^{\infty}\left[r_k(\Delta y_k)^2-c_ky_{k+1}^2\right]<0.$$

On the other hand, to prove nonoscillation of (9) we need to show that there exists $N\in\mathbb{N}$ such that for every nontrivial $y\in\mathcal{D}(N)$ we have $\mathcal{F}_d(y;N,\infty)>0$.

A typical example of the oscillation criterion proved using the variational method is the discrete version of the Leighton-Wintner oscillation criterion.

Theorem 2. *Suppose that $r_k > 0$ for large k and*

$$\sum^{\infty} r_k^{-1} = \infty = \sum^{\infty} c_k. \tag{12}$$

Then equation (9) *is oscillatory.*

Proof. Let $N \in \mathbb{N}$ be arbitrary. Define for $N < n < m < M$ (which will be determined later) a sequence $y \in \mathcal{D}(N)$ as follows

$$y_k = \begin{cases} \left(\sum_{j=N}^{k-1} r_j^{-1}\right)\left(\sum_{j=N}^{n-1} r_j^{-1}\right)^{-1}, & N+1 \le k \le n, \\ 1, & n+1 \le k \le m-1, \\ \left(\sum_{j=k}^{M-1} r_j^{-1}\right)\left(\sum_{j=m}^{M-1} r_j^{-1}\right)^{-1}, & m \le k \le M-1, \\ 0, & k \ge M. \end{cases}$$

Then we have

$$\begin{aligned} \mathcal{F}_d(y; N, \infty) &= \sum_{k=N}^{\infty} \left[r_k(\Delta y_k)^2 - c_k y_{k+1}^2\right] = \sum_{k=N}^{M-1} \left[r_k(\Delta y_k)^2 - c_k y_{k+1}^2\right] \\ &= \left(\sum_{k=N}^{n-1} + \sum_{k=n}^{m-1} + \sum_{k=m}^{M-1}\right)\left[r_k(\Delta y_k)^2 - c_k y_{k+1}^2\right] \\ &= \left(\sum_{k=N}^{n-1} r_k^{-1}\right)^{-1} - \sum_{k=N}^{n-1} c_k y_{k+1}^2 - \sum_{k=n}^{m-2} c_k - \sum_{k=m-1}^{M-1} c_k y_{k+1}^2 + \left(\sum_{k=m}^{M-1} r_k^{-1}\right)^{-1}. \end{aligned}$$

Now, using the discrete version of the second mean value theorem of the sum calculus (see, e.g. [11]), there exists $\tilde{m} \in [m-1, M-1]$ such that

$$\sum_{k=m-1}^{M-1} c_k y_{k+1}^2 \le \sum_{k=m-1}^{\tilde{m}} c_k.$$

Let $n > N$ be fixed. Since (12) holds, for every $\varepsilon > 0$ there exist $M > m > n$ such that $\sum_{k=n}^{\tilde{m}} c_k > \mathcal{F}_d(y; N, n-1) + \varepsilon$ whenever $\tilde{m} > m$ and $\left(\sum_{k=m}^{M-1} r_k^{-1}\right)^{-1} < \varepsilon$. Consequently, we have

$$\mathcal{F}_d(y; N, \infty) = \mathcal{F}_d(y; N, n-1) - \sum_{k=n}^{\tilde{m}} c_k + \left(\sum_{k=m}^{M-1} r_k^{-1}\right)^{-1} < 0$$

what we needed to prove.

A more sophisticated application of the construction of the sequence y leads to a discrete versions of Nehari-type oscillation criteria, for more details we refer to

[11], where the variational method is used to derive oscillation criteria for $2n$-order Sturm-Liouville difference equations.

In proving nonoscillation criteria using the variational method, the following discrete version of the Wirtinger-type inequality is a very useful tool, see [23].

Theorem 3. *Let M_k be a positive sequence such that $\Delta M_k \neq 0$ for $k \geq N$. Then for every $y \in \mathcal{D}(N)$ we have*

$$\sum_{k=N}^{\infty} |\Delta M_k| y_{k+1}^2 \leq \psi_N \sum_{k=N}^{\infty} \frac{M_k M_{k+1}}{|\Delta M_k|} (\Delta y_k)^2,$$

where

$$\psi_N := \left(\sup_{k \geq N} \frac{M_k}{M_{k+1}} \right) \left\{ 1 + \left(\sup_{k \geq N} \frac{|\Delta M_k|}{|\Delta M_{k-1}|} \right)^{1/2} \right\}^2 . \tag{13}$$

A typical example of the application of the Wirtinger inequality is the next Nehari-type nonoscillation criterion which is proved for higher order equations in [23].

Theorem 4. *Suppose that there exists a positive sequence M_k such that ΔM_k is eventually nonzero and satisfies $0 < \psi := \limsup_{N \to \infty} \psi_N < \infty$, where ψ_N is defined by* (13)*. If*

$$\limsup_{k \to \infty} \frac{1}{M_k} \sum_{j=k}^{\infty} c_j^+ < \frac{1}{\psi}, \quad c_k^+ := \max\{0, c_k\},$$

then equation (9) *is nonoscillatory.*

We finish this section with a Hille-Nehari type nonoscillation criterion proved using the Riccati technique. This criterion is presented in [16] for the half-linear second order difference equation

$$\Delta(r_k \Phi(\Delta x_k)) + c_k \Phi(x_{k+1}) = 0, \quad \Phi(x) := |x|^{p-2} x, \ p > 1,$$

but for the sake of simplicity we formulate it for linear equation (9).

Observe that according to the Sturm comparison theorem for (9), to prove nonoscillation of (9), it actually suffices to find $N \in \mathbb{N}$ and a sequence w_k defined for $k \geq N$, satisfying $w_k + r_k$ and the *inequality*

$$\Delta w_k + c_k + \frac{w_k^2}{w_k + r_k} \leq 0. \tag{14}$$

Theorem 5. *Suppose that $r_k > 0$ for large k, $\sum^{\infty} c_k$ is convergent and*

$$\lim_{k \to \infty} \frac{r_k^{-1}}{\sum^{k-1} r_j^{-1}} = 0. \tag{15}$$

If

$$\limsup_{k\to\infty}\left(\sum^{k-1} r_j^{-1}\right)\left(\sum_{j=k}^{\infty} c_j\right) < \frac{1}{4}, \quad \liminf_{k\to\infty}\left(\sum^{k-1} r_j^{-1}\right)\left(\sum_{j=k}^{\infty} c_j\right) > -\frac{3}{4} \tag{16}$$

then (9) *is nonoscillatory.*

Note that assumption (15) has no analogue in the continuous version of Theorem 5 (see e.g. [12]) and necessity of this assumption in Theorem 5 is caused by the term $r_k + w_k$ in the denominator of the last term in (10). We define the sequence

$$w_k := \frac{1}{4}\left(\sum^{k-1} r_j^{-1}\right)^{-1} + \sum_{j=k}^{\infty} c_j$$

and in order to show that (16) imply that w is a solution of (14) satisfying $w_k + r_k > 0$ we need just assumption (15). In the continuous case, the denominator of the last term in the Riccati equation (5) is r, i.e. does not contain the function w and no analogue of (15) is needed in the continuous modification of this proof.

Finally note that the oscillation theory of (9) is now deeply developed and many oscillation and nonoscillation criteria for (2) have their continuous counterparts, see e.g. [1, Chap. VI].

3. Transformation and oscillation theory of symplectic difference systems

Denote $u_k = r_k \Delta x_k$ in (9). Then we can write this equation as the 2-dimensional first order system

$$\Delta\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} 0 & r_k^{-1} \\ -c_k & 0 \end{pmatrix}\begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix} \tag{17}$$

and expanding the difference operator as recurrence system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \mathcal{S}_k \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad \mathcal{S}_k := \begin{pmatrix} 1 & \frac{1}{r_k} \\ -\frac{c_k}{r_k} & 1 - \frac{c_k}{r_k} \end{pmatrix}.$$

By a direct computation it is not difficult to verify that the matrix in the last system is symplectic, i.e., it satisfies the identity $\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}$, $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Consider now the general $2n \times 2n$ symplectic difference system

$$z_{k+1} = \mathcal{S}_k z_k, \tag{18}$$

where $z = \binom{x}{u}$, $\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}$ is a symplectic matrix, i.e., it satisfies

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

$x, u \in \mathbb{R}^n$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{R}^{n\times n}$. Symplectic difference systems cover a large variety of difference equations and systems. For example, the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k$$

with symmetric $n \times n$ matrices B, C and the matrix $(I - A)$ invertible is a special case of (18), see [2]. Since the $2n$-order Sturm-Liouville equation

$$\sum_{\nu=0}^{n} \Delta^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) = 0, \quad \Delta^\nu := \Delta(\Delta^{\nu-1}) \tag{19}$$

can be written as (3) with special matrices A, B, C (see, e.g. [2]), symplectic difference systems cover Sturm-Liouville equations as well.

Let $Z = \binom{X}{U}$, $\bar{Z} = \binom{\bar{X}}{\bar{U}}$ be $2n \times n$ solutions of (18), then $\Delta(Z_k^T \mathcal{J} \bar{Z}_k) = 0$, i.e., $Z_k^T \mathcal{J} \bar{Z}_k = \mathcal{M}$, where $\mathcal{M}$ is a constant $n \times n$ matrix. This identity can be regarded as the extension of the classical Casoratian identity to (18). If $\bar{Z} = Z$, $\mathcal{M} = 0$ and rank $Z_k = n$, then Z is called a *conjoined basis* of (18). Oscillatory properties of solutions of (18) are defined using the concept of a *focal point* in the same way as oscillatory properties of (9) via the concept of generalized zero.

Recall that an interval $(m, m+1]$ contains a *focal point* of a $2n \times n$ solution $Z = \binom{X}{U}$ of (18) if

$$\operatorname{Ker} X_{m+1} \subseteq \operatorname{Ker} X_m \quad \text{and} \quad D_m := X_m X_{m+1}^\dagger \mathcal{B}_m \not\geq 0$$

fail to hold. Here Ker, † and $\geq$ mean kernel, Moore-Penrose generalized inverse and nonnegative definiteness of the matrix indicated.

Let $\mathcal{R}_k = \begin{pmatrix} H_k & M_k \\ K_k & N_k \end{pmatrix}$ be symplectic $2n \times 2n$ matrices (H, K, M, N being $n \times n$ matrices) and consider the transformation

$$z_k = \mathcal{R}_k \tilde{z}_k. \tag{20}$$

This transformation transforms (18) into the system $\tilde{z}_{k+1} = \tilde{\mathcal{S}}_k \tilde{z}_k$, $\tilde{\mathcal{S}}_k = \mathcal{R}_{k+1}^{-1} \mathcal{S}_k \mathcal{R}_k$ and this new system is again symplectic as can be verified by a direct computation. Moreover, if $M_k \equiv 0$ in $\mathcal{R}_k$, then transformation (20) preserves focal points of transformed systems and hence also their oscillatory behavior as it is shown in [6]. In that paper the Roundabout theorem for (18) is presented, in particular, it is proved that the quadratic functional

$$\mathcal{F}(z) := \sum_{k=0}^{N} z_k^T \{\mathcal{S}_k^T \mathcal{K} \mathcal{S}_k - \mathcal{K}\} z_k, \quad \mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

over the class of sequences satisfying $\mathcal{K} z_{k+1} = \mathcal{K} \mathcal{S}_k z_k$, $\mathcal{K} z_0 = 0 = \mathcal{K} z_{N+1}$, and the Riccati matrix difference equation

$$Q_{k+1} = (\mathcal{C}_k + \mathcal{D} Q_k)(\mathcal{A}_k + \mathcal{B} Q_k)^{-1}$$

play the same role as (8) and (10) in the oscillation theory of (9).

In the remaining part of this section we present two particular transformations of (18) where the so-called *trigonometric difference system* appears. A trigonometric difference systems (introduced by Anderson [3]) is the symplectic difference system whose matrix satisfies the additional condition $\mathcal{J}^T \mathcal{S}_k \mathcal{J} = \mathcal{S}_k$. This means that transformation (20) with $\mathcal{R}_k = \mathcal{J}$ (the so-called *reciprocity transformation*, see [6]) transforms system (18) into itself. Hence, trigonometric system can be written in the form

$$\begin{pmatrix} s_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_k & \mathcal{Q}_k \\ -\mathcal{Q}_k & P_k \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \tag{21}$$

where the matrices $\mathcal{P}, \mathcal{Q}$ satisfy the identities

$$\mathcal{P}_k^T \mathcal{Q}_k = \mathcal{Q}_k^T \mathcal{P}_k, \quad \mathcal{P}_k^T \mathcal{P}_k + \mathcal{Q}_k^T \mathcal{Q}_k = I. \tag{22}$$

In particular, if $n = 1$, then (22) implies the existence of $\varphi_k \in [0, 2\pi)$ such that

$$\sin \varphi_k = \mathcal{Q}_k, \quad \cos \varphi_k = \mathcal{P}_k \tag{23}$$

and then

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix} = \begin{pmatrix} \sin \left(\sum^{k-1} \varphi_j \right) \\ \cos \left(\sum^{k-1} \varphi_j \right) \end{pmatrix}, \quad \begin{pmatrix} c_k \\ -s_k \end{pmatrix} = \begin{pmatrix} \cos \left(\sum^{k-1} \varphi_j \right) \\ \sin \left(\sum^{k-1} \varphi_j \right) \end{pmatrix}$$

form the basis of the solution solution space of (21).

Theorem 6. (Trigonometric transformation, [7]) *There exist $n \times n$ matrices H and K such that H is nonsingular, $H^T K = K^T H$, and the transformation*

$$\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} H^{-1} & 0 \\ -K^T & H^T \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{24}$$

transforms the symplectic system (18) *into trigonometric system* (21) *without changing the oscillatory behavior. Moreover, the matrices $\mathcal{P}$ and $\mathcal{Q}$ from* (21) *may be explicitly given by*

$$\mathcal{P}_k = H_{k+1}^{-1}(\mathcal{A}_k H_k + \mathcal{B}_k K_k) \quad \text{and} \quad \mathcal{Q}_k = H_{k+1}^{-1} \mathcal{B}_k H_k^{T-1}. \tag{25}$$

The previous statement is a discrete version of the trigonometric transformation of linear Hamiltonian *differential* systems established in [10], where it is proved that any linear Hamiltonian differential system

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u \tag{26}$$

with B, C symmetric, can be transformed by a transformation preserving oscillatory nature of transformed systems into the trigonometric differential system

$$s' = Q(t)c, \quad c' = -Q(t)s \tag{27}$$

with a symmetric matrix Q. The terminology trigonometric system is again justified by the scalar case $n=1$ since $\sin\left(\int^t Q(s)\,ds\right), \cos\left(\int^t Q(s)\,ds\right)$ is a solution of this system. It is known (see [26, Chap. VII] that (27) with $Q(t)\geq 0$ is oscillatory (i.e., there exists a conjoined basis $\binom{S}{C}$ and a sequence $t_n\to\infty$ such that $\det S(t_n)=0$) if and only if $\int^\infty \operatorname{Tr} Q(t)\,dt=\infty$, Tr stands for the trace of the matrix indicated. In the discrete case a necessary and sufficient condition for oscillation of (21) is known only in case when $\mathcal{Q}$ is nonsingular and reads

$$\sum^{\infty} \operatorname{arccotg} \lambda^{[1]}\left(\mathcal{Q}_k^{-1}\mathcal{P}_k\right)=\infty,$$

$\lambda^{[1]}(\cdot)$ denotes the least eigenvalue of the matrix indicated, see [7]. Since the matrix $\mathcal{Q}$ is given by (25), nonsingularity of $\mathcal{Q}$ is equivalent to nonsingularity of $\mathcal{B}$. However, symplectic systems with $\mathcal{B}$ nonsingular do not cover many important cases, e.g. the higher order Sturm-Liouville equation (19). For this reason it would be very useful to know a necessary and sufficient condition for oscillation of (21) also in the case when Q is allowed to be singular.

We finish this section with a discrete version of the Prüfer transformation.

Theorem 7. ([8]) *Let $Z=\binom{X}{U}$ be a $2n\times n$ matrix conjoined basis of* (18). *Then there exist nonsingular $n\times n$ matrix H and $n\times n$ matrices S, C such that $\binom{X}{U}$ can be expressed in the form*

$$X_k=S_k^T H_k,\quad U_k=C_k^T H_k, \tag{28}$$

where $\binom{S}{C}$ is a solution of the trigonometric system (21) *satisfying $S_k^T S_k+C_k^T C_k=I$, $S_k^T C_k-C_k^T S_k=0$. The matrices $\mathcal{P},\mathcal{Q}$ are given by the formulas*

$$\begin{aligned}\mathcal{P}&=(H_{k+1}^T)^{-1}\tbinom{X_k}{U_k}^T\mathcal{S}_k^T\tbinom{X_k}{U_k}H_k^{-1}-\Delta H_k,\\ \mathcal{Q}&=(H_{k+1}^T)^{-1}\tbinom{X_k}{U_k}^T\mathcal{S}_k^T\mathcal{J}\tbinom{X_k}{U_k}H_k^{-1}\end{aligned}$$

and H solves the first order system

$$\Delta H_k=(\tilde Z_{k+1})^T(\mathcal{S}_k\tilde Z_k-\Delta\tilde Z_k)H_k,\quad \tilde Z=\begin{pmatrix}S^T\\ C^T\end{pmatrix}.$$

In the continuous case, the Prüfer transformation for linear Hamiltonian differential systems (26) was established in [4] as a matrix extension of the classical Prüfer transformation for (2) proved in [25]. If $n=1$ in Theorem 7 and (18) is rewritten Sturm-Liouville equation (9) (compare (17)), then (28) reduces to

$$x_k=H_k\sin\left(\sum^{k-1}\varphi_j\right),\quad r_k\Delta x_k=H_k\cos\left(\sum^{k-1}\varphi_j\right),$$

where φ_k is given by (23), and Theorem 7 is really a discrete version of the classical Prüfer transformation.

4. Higher order linear difference equations

Consider the n-th order linear difference equation

$$L(y)_k := x_{k+n} + a_k^{[n-1]} x_{k+n-1} + \dots a_k^{[1]} x_{k+1} + a_k^{[0]} x_k = 0. \tag{29}$$

Basic facts of the qualitative theory of (29) can be found in [1,17]. One of the motivation for the investigation of oscillatory properties of linear differential and difference equations is the so-called Polya factorization. In the continuous case this problem was resolved in [24] (see also [9]) and in the discrete case it is treated in the fundamental paper of Hartman [21]. Recall now some statements of that paper. An integer $k+m$ is said to be the *generalized zero point of multiplicity* m of a sequence x_k if $x_k \neq 0$, $x_{k+1} = \dots = x_{k+m-1} = 0$ and $(-1)^{m-1} x_{k+m} x_k \leq 0$. If $m = 1$ and $n = 2$ then this definition complies with the definition of the generalized zero of (9) with $r_k \equiv 1$. Observe also that a nontrivial solution of linear equation (29) cannot have a generalized zero of multiplicity greater than $n-1$ as can be verified by a direct computation. Equation (29) is said to be *disconjugate* on the interval $[0, N]$ if every nontrivial solution has at most $n-1$ generalized zeros (counting multiplicity) in $[0, M+n]$ and the solutions satisfying $x_0 = \dots = x_j = 0$, $x_{j+1} \neq 0$, $j \in \{0, \dots, n-2\}$ have at most $n-j-2$ generalized zeros (again counting multiplicity) in $(j+1, N+n-j-1]$.

Theorem 8. *Suppose that* (29) *is disconjugate on* $[0, N]$. *Then there exists a fundamental system of solutions of this equation* $x^{[1]}, \dots, x^{[n]}$ *such that* n *Casoratians*

$$C(x^{[1]}, \dots, x^{[j]})_k := \begin{vmatrix} x_k^{[1]} & \dots\ x_k^{[j]} \\ \vdots & \vdots \\ x_{k+j-1}^{[1]} & \dots\ x_{k+j-1}^{[j]} \end{vmatrix} > 0$$

for $k \in [0, N]$ *and* $j = 1, \dots, n$. *Moreover, the operator* L *admits Polya's factorization*

$$L(y)_k = \alpha_k^{[0]} \alpha_k^{[1]} \cdots \alpha_k^{[n-1]} \Delta \left\{ \frac{1}{\alpha_k^{[n-1]}} \Delta \left[\dots \Delta \left(\frac{y_k}{\alpha_k^{[0]}} \right) \dots \right] \right\}, \tag{30}$$

where (for $j = 2, \dots, n-1$*)*

$$\alpha_k^{[0]} = x_k^{[1]},\ \alpha_k^{[1]} = \Delta \left(\frac{x_k^{[2]}}{x_k^{[1]}} \right),\ \alpha_k^{[j]} = \frac{C(x^{[1]}, \dots, x^{[j-1]})_{k+1} C(x^{[1]}, \dots, x^{[j+1]})_k}{C(x^{[1]}, \dots, x^{[j]})_{k+1} C(x^{[1]}, \dots, x^{[j]})_k} \tag{31}$$

Another important statement concerning Polya's factorization is the so-called Trench canonical factorization, see [20].

Theorem 9. *Suppose that* (29) *is eventually disconjugate, i.e., there exists* $N \in \mathbb{N}$ *such that this equation is disconjugate on* $[N, M]$ *for every* $M > N$. *Then* *the operator L can be expressed on $[N,\infty)$ in the form* (30) *with the sequences* $\alpha^{[1]},\dots,\alpha^{[n-1]}$ *satisfying*

$$\sum^{\infty}\alpha_k^{[j]}=\infty,\quad j=1,\dots,n-1.$$

Recall that the canonical factorization for disconjugate linear *differential operators* was established by Trench [27] and that disconjugate linear differential operators have many properties similar to those of the simple operator of the n-th derivative $\tilde{L}(y):=y^{(n)}$, see e.g. [18]. This book also represent a good motivation for discretization of continuous results.

Now let us turn out attention to the higher order, two-term, Sturm-Liouville equation

$$(-1)^n\Delta^n(r_k\Delta^n y_k)=q_k y_{k+n} \tag{32}$$

with $r_k\neq 0$. The most of the next results can be extended to the general equation (19), but to see better the similarity between the second order case (9) and higher order equations, we consider two-term equation (32) only. Since this equation can be written as a linear Hamiltonian difference system and hence also as a symplectic difference system (18), oscillatory properties of (32) are defined via those of the corresponding symplectic difference system. Denote

$$\mathcal{D}_n(N)=\{y=\{y_k\}_{k=N}^{\infty}:\ y_N=\ldots=y_{N+n-1}=0,\ \exists M>N+n-1,\ y_k=0,\ k\geq M\}$$

(observe that the class of sequences $\mathcal{D}(N)$ defined in Section 2 coincides with $\mathcal{D}_1(N)$). The quadratic functional associated with (32) is

$$\mathcal{F}(y;N,\infty)=\sum_{k=N}^{\infty}\left[r_k(\Delta^n y_k)^2-q_k y_{k+n}^2\right]$$

and equation (32) is nonoscillatory if and only if there exists $N\in\mathbb{N}$ such that $\mathcal{F}(y;N,\infty)>0$ for every nontrivial $y\in\mathcal{D}_n(N)$. This statement is a direct extension of the of the variational oscillation method for second order equations to (32). Using a modified construction from Section 2, one can prove the following higher order extension of the Leighton-Wintner criterion given in Theorem 2.

Theorem 10. ([11]) *Suppose that $r_k>0$ for large k, $\sum^{\infty} r_k^{-1}=\infty$ and there exists $j\in\{0,\dots,n-1\}$ such that $\sum^{\infty} q_k k^{(j)}=\infty$, where $k^{(j)}:=k(k-1)\cdots(k-j+1)$, $k^{(0)}=1$ is the so-called generalized j-th power. Then equation* (32) *is oscillatory.*

Concerning a higher order extension of the Hille-Nehari-type nonoscillation criterion given in Theorem 5, the proof of this extension is essentially the same as those of Theorem 5, only one has to apply the Wirtinger inequality n-times (instead of once as in Theorem 5). We do not formulate the result explicitly, but we refer to the recent papers [13,23].

In Theorem 10 and also in its nonoscillatory counterpart given in [13], equation (32) is viewed in a certain sense as a perturbation of the one-term (nonoscillatory) equation $(-1)^n \Delta^n (r_k \Delta^n y_k)^{(n)} = 0$ and it is shown that if the sequence q_k is "sufficiently positive", i.e., $\sum^\infty q_k k^{(j)} = \infty$, ("not too positive") then (32) becomes oscillatory (remains nonoscillatory).

To formulate an open problem connected with (32), consider the $2n$-order Sturm-Liouville *differential* equation

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} = q(t)y, \tag{33}$$

where $\alpha \notin \{1, 3, \dots, 2n-1\}$ is a real constant. A typical approach when investigating oscillatory properties of (33) used e.g. in [14,15], is that this equation is not viewed as a perturbation of the one-term equation $(-1)^n (t^\alpha y^{(n)}) = 0$, but as a perturbation of the Euler-type equation

$$(-1)^n (t^\alpha y^{(n)}) + \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = 0 \tag{34}$$

$\gamma_{n,\alpha} = (-4)^{-n} \prod_{i=0}^{n-1} (2n - \alpha - 2i - 1)(2n + \alpha - 2i - 1)$ being the so-called critical oscillation constant. In the discrete case we also have in disposal an Euler-type equation

$$(-1)^n \Delta^{2n} x_k + \frac{\gamma}{(k + 2n - 1)^{(2n)}} x_k = 0 \tag{35}$$

whose solutions are of the form $x_k = \frac{\Gamma(\lambda+k)}{\Gamma(k)}$, $\Gamma(t) = \int_0^\infty \mathrm{e}^{-s} t^{s-1}\, ds$ being the classical Γ function, and λ is a solution of the characteristic equation $(-1)^n \lambda(\lambda - 1) \cdots (\lambda - 2n + 1) + \gamma = 0$, see [1, Chap. III]. However, equation (35) (in contrast to (34)) *is not in self-adjoint form*, since the second term on left-hand-side of this equation contains x with index k instead of $k + n$ (compare (32)). Hence the above mentioned "continuous" idea cannot be directly applied to difference equations.This suggests the following open problem; to find a two-term *self-adjoint nonoscillatory* difference equation which can be solved explicitly (like (34) in the continuous case) and to use this equation as "perturbation equation" in the oscillation theory of (32).

Acknowledgement. Research supported by the Grant No. 201/98/0677 of the Grant Agency of Czech Republic.

References

1. R. P. Agarwal, Difference Equations and Inequalities: Theory, Methods and Applications, Second Edition, Pure and Applied Mathematics, M. Dekker, New York - Basel - Hong Kong, 2000.
2. C. D. Ahlbrandt, A. C. Peterson, Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations Kluwer Academic Publishers, Boston, 1996.

3. D. Anderson, *Discrete trigonometric matrix functions*, Panamer. Math. J. **7** (1997), 39-54.
4. J. H. Barrett, *A Prüfer transformation for matrix differential equations*, Proc. Amer. Math. Soc. **8** (1957), 510-518.
5. M. Bohner, *Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions*, J. Math. Anal. Appl. **199** (1996), 804–826.
6. M. Bohner, O. Došlý, *Disconjugacy and transformations for symplectic systems*, Rocky Mountain J. Math. **27** (1997), 707–743.
7. M. Bohner, O. Došlý, *Trigonometric transformation of symplectic difference systems*, J. Differential Equ. **163** (2000), 113-129.
8. M. Bohner, O. Došlý, *Discrete Prüfer transformation*, to appear in Proc. Amer. Math. Soc.
9. W. A. Coppel, Disconjugacy, Lectures Notes in Math., No. 220, Springer Verlag, Berlin-Heidelberg 1971.
10. O. Došlý, *On transformations of self-adjoint differential systems and their reciprocals*, Ann. Polon. Math. **50** (1990), 223-234.
11. O. Došlý, *Oscillation criteria for higher order Sturm-Liouville difference equations*, J. Differ. Equations Appl. **4** (1998), 425-450.
12. O. Došl y, *Methods of oscillation theory of half-linear second order differential equations*, Czech. Math. J. **50 (125)** (2000), 657-671.
13. O. Došlý, R. Hilscher, *A class of Sturm-Liouville difference equations: (non)oscillation constants and property BD*, submitted.
14. O. Došlý, J. Osička, *Kneser-type oscillation criteria for self-adjoint, two term, differential equations*, Georgian J. Math., **2** (1995), 241-258.
15. O. Došlý, J. Osička, *Oscillation and nonoscillation of higher order self-adjoint differential equations*, to appear in Czech. Math. J.
16. O. Došlý, P. Řehák, *Nonoscillation criteria for half-linear second order difference equations*, to appear in Comput. Appl. Math.
17. S. N. Elaydi, An Introduction to Difference Equations, Second Edition, Springer Verlag, 2000.
18. U. Elias, Oscillation Theory of Two-Term Differential equations, Kluwer, Dordrecht-Boston-London, 1997.
19. I. M. Gelfand, S. V. Fomin, Calculus of Variations, Prentice Hall, Engelwood, 1963.
20. B. Harris, R. J. Kruger, W. T.Trench, *Trench's canonical form for a disconjugate n-th order linear difference equations*, Panamer. Math. J. **8** (1998), 55-71.
21. P. Hartman, *Difference equations: disconjugacy, principal solutions, Green's function, complete monotonicity*, Trans. Amer. Math. Soc. **246** (1978), 1–30.
22. W. G. Kelley, A. Peterson, Difference Equations: An Introduction with Applications, Acad. Press, San Diego, 1991.
23. S. Peña *Discrete spectra criteria for singular difference operators*, Math. Bohem. **124** (1999), 35–44.
24. G. Polya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1924), 312-324.
25. H. Prüfer, *Neue Herleitung der Sturm-Liouvilleschen Reihenentwicklung stetiger Funktionen*, Math. Ann. **95** (1926), 499-518.
26. W. T. Reid, Sturmian Theory for Ordinary Differential Equations, Springer Verlag, New York-Heidelberg-Berlin 1980.
27. W. F. Trench, *Canonical forms and principal systems of general disconjugate equations*, Trans. Amer. Math. Soc. **189** (1974), 319-327.