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EXISTENCE OF POSITIVE SOLUTIONS OF N-DIMENSIONAL SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS ENTERING INTO A SINGULAR POINT

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Abstract. An n-dimensional system of nonlinear differential equations is considered. It is shown that a singular initial problem has at least one solution (or infinitely many solutions) with positive coordinates. Moreover, asymptotic behaviour of these solutions is described by means of the curves that are defined implicitly.

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1. Introduction

This paper deals with the existence of solutions of the singular initial problem, stated for n-dimensional nonlinear systems of differential equations, entering into a singular point. Namely, we will consider the initial problems ($S_\pm$), (IP) where

$$g_i(x)\, y_i' = \pm \left[\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x) \right], \quad i = 1, 2, \dots, n, \tag{$S_\pm$}$$

$$y_i(0^+) = 0, \quad i = 1, 2, \dots, n. \tag{IP}$$

Let us denote as I_{x_0} an interval of the form $I_{x_0} = (0, x_0]$ and I_{y_0} an interval of the form $I_{y_0} = (0, y_0]$ with $x_0, y_0 > 0$. The systems ($S_\pm$) will be considered under the following main assumptions:

(C1) $g_i \in C(I_{x_0}, \mathbb{R}^+),\ i = 1, 2, \dots, n,\ \mathbb{R}^+ = (0, \infty)$;

(C2) $\alpha_i \in C^1(I_{y_0}, \mathbb{R}),\ \alpha_i > 0$ on I_{y_0}, $\alpha_i' > 0$ on $I_{y_0}, \alpha_i(0^+) = 0,\ i = 1, 2, \dots, n$;

(C3) $\omega_i \in C^1(I_{x_0}, \mathbb{R}),\ \omega_i > 0$ on I_{x_0}, $\omega_i' > 0$ on $I_{x_0}, \omega_i(0^+) = 0,\ i = 1, 2, \dots, n$;

(C4) $a_{ij} = \text{const},\ i, j = 1, 2, \dots, n;\ a_{ii} > 0, a_{ij} \le 0,\ i, j = 1, 2, \dots, n,\ i \ne j,\ \Delta = \det A > 0,\ A = (a_{ij})_{i,j=1}^n$, and cofactors $C_{ij} = (-1)^{i+j} A_{ij} \ge 0$ where A_{ij} are minors of the elements a_{ij} of the matrix A.

We will consider the systems ($S_\pm$) in the domain $Q \equiv I_{x_0} \times \underbrace{I_{y_0} \times \cdots \times I_{y_0}}_{n}$, i.e. the corresponding results will concern the existence of solutions of this problem having positive coordinates. More precisely, we define a solution of the problems ($S_\pm$), (IP) in the sense of the following definition:

Definition 1. A function $y = y(x) = (y_1(x), \dots, y_n(x)) \in C^1(I_{x^*}, \mathbb{R}^n)$ with $0 < x^* \le x_0$ is said to be a *solution of the singular problem* (S_+), (IP) (or (S_-), (IP)) on interval I_{x^*} if:

1) $(x, y(x)) \in Q$ for $x \in I_{x^*}$;
2) y satisfies (S_+) (or (S_-)) on I_{x^*};
3) $y_i(0^+) = 0,\ i = 1, 2, \dots, n$.

The origin of coordinates $O = (0, 0, \dots, 0)$ is a boundary point of introduced domain Q. The possibility $g_i(0^+) = 0,\ i = 1, 2, \dots, n$ is not excluded from our investigation (note that, for validity of assumptions of the theorems formulated below, this condition is often tacitly assumed). So the problems ($S_\pm$), (IP), in view of assumptions (C1)–(C4), are really the singular problems and known theorems about existence of solution of initial problems cannot be used.

Various initial singular problems for ordinary differential equations were widely considered (let us cite e.g. the works of K. Balla [1], J. Baštinec and J. Diblík [2], J. Diblík [3]–[5], I.T. Kiguradze [11], N.B. Konyukhova [12], Chr. Nowak [13], D. O'Regan [14], M. Růžičková [15]), namely after the appearance of the pioneering work of V.A. Chechyk [10], the solvability of the considered problems ($S_\pm$), (IP) cannot be established by using the results which are known to the authors of this paper.

Let us explain the scheme of our investigation. We consider the implicit system of nondifferential equations with respect to unknowns $z_1, \dots, z_n$ which arise if in the systems ($S_\pm$) the left-hand sides equal zero (i.e. if $g_i(x) \equiv 0, i = 1, 2, \dots, n$):

$$\sum_{j=1}^{n} a_{ij}\alpha_j(z_j) = \omega_i(x), \quad i = 1, 2, \dots, n. \tag{1}$$

Considering this system we conclude that it is equivalent to the system consisting of separated scalar equations:

$$\alpha(z) = \Omega(x) \equiv A^{-1}\omega(x)$$

with $\alpha(z) = (\alpha_1(z_1), \dots, \alpha_n(z_n))^T$, $\Omega = (\Omega_1, \dots, \Omega_n)^T$, $\omega = (\omega_1, \dots, \omega_n)^T$ or

$$\alpha_i(z_i) = \Omega_i(x), \quad i = 1, 2, \dots, n$$

with (see (C4))

$$\Omega_i(x) \equiv \frac{1}{\Delta} \cdot \sum_{j=1}^{n} C_{ji}\omega_j(x), \;\; i = 1, 2, \dots, n. \tag{2}$$

Note that in the view of our assumptions $\Omega_i(x) > 0$ on I_{x_0} and $\Omega_i(0^+) = 0$, $i = 1, 2, \dots, n$. Solving these scalar equations with respect to z_i, $i = 1, 2, \dots, n$ we get

$$z_i = \alpha_i^{-1}[\Omega_i(x)], \quad i = 1, 2, \dots, n,$$

where α_i^{-1} is the inverse function of the function α_i (existence of it follows from the condition (C2)).

It can be expected that under appropriate conditions the asymptotic behaviour of a solution $y(x) = (y_1(x), \dots, y_n(x))$ of the problem (S_+), (IP) (or (S_-), (IP)) for $x \to 0^+$ will be in a sense similar to the asymptotic behaviour of $z(x) = (z_1(x), \dots, z_n(x))$ for $x \to 0^+$, i.e. it can be expected that the asymptotic formulae

$$y_i(x) \approx z_i(x) \quad \text{if} \quad x \to 0^+, \quad i = 1, 2, \dots, n$$

will hold.

The proofs of Theorem 1 and Theorem 2 below are based on known qualitative properties of solutions of differential equations. Besides, in the proof of Theorem 2 the topological method of T. Ważewski is used (see, e.g., [9], [16]). Except this, properties of functions that are defined implicitly are applied in these proofs. Let us note that one of the advantages of our results is the fact that although properties of implicit functions are used, the assumptions of them are easily verifiable and do not use any supposition which cannot be verified immediately. Moreover, it is easy to get corresponding linear cases as a consequence of our results. Results of this paper generalize previous ones, given in the work [8].

2. Auxiliary Lemmas

Let us state a lemma on existence and differentiability of a function given implicitly by the equation

$$\tilde{\alpha}(y) = \tilde{\omega}(x) \qquad \text{if} \quad (x, y) \in I_{x_0} \times I_{y_0}. \tag{3}$$

Lemma 1. *Let the following assumptions be valid:*

$$\tilde{\alpha} \in C^1(I_{y_0}, \mathbb{R}),\ \tilde{\alpha} > 0 \text{ on } I_{y_0},\ \tilde{\alpha}' > 0 \text{ on } I_{y_0} \text{ and } \tilde{\alpha}(0^+) = 0;$$
$$\tilde{\omega} \in C^1(I_{x_0}, \mathbb{R}),\ \tilde{\omega} > 0 \text{ on } I_{x_0},\ \tilde{\omega}' > 0 \text{ on } I_{x_0} \text{ and } \tilde{\omega}(0^+) = 0.$$

Then there exists a unique solution

$$y = \varphi(x) \equiv \tilde{\alpha}^{-1}[\tilde{\omega}(x)]$$

of equation (3) *on an interval* $I_{\delta_0} \subset I_{x_0}$ *with properties:*

$$\varphi \in C^1(I_{\delta_0}, \mathbb{R}),\ \ \varphi \in I_{y_0} \quad \textit{and} \quad \varphi' > 0 \quad \textit{on} \quad I_{\delta_0};$$
$$\varphi(0^+) = 0;$$
$$\varphi'(x) \equiv \frac{\tilde{\omega}'(x)}{\tilde{\alpha}'[\varphi(x)]}\,, \ \ x \in I_{\delta_0}.$$

The proof of Lemma 1 can be made in an elementary way and is therefore omitted.

Remark 1. The next obvious property will be used in the sequel: let $\varepsilon_1, \varepsilon_2$ be two positive constants and $\varepsilon_1 < \varepsilon_2$. Then there exists an interval $I_{\delta_1} \subset I_{\delta_0}$ (determined by the requirement $\varphi(\varepsilon_2 x) \le y_0$ on I_{δ_1}, i.e. $\delta_1 = \min\{\delta_0 \varepsilon_2^{-1}, \delta_0\}$) such that the inequality $\varphi(\varepsilon_1 x) < \varphi(\varepsilon_2 x)$ holds on I_{δ_1}.

Lemma 2. *Let all assumptions of Lemma* 1 *be valid and, moreover, there exist a constant* $M \in \mathbb{R}^+$ *such that*

$$\tilde{\alpha}(y) \le M \tilde{\alpha}'(y), \quad y \in I_{y_0}.$$

Then the unique solution $y = \varphi(x)$ *of equation* (3) *defined on an interval* $I_{\delta_0} \subset I_{x_0}$ *satisfies the inequality:*

$$\varphi'(x) \le M \cdot \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)}, \ \ x \in I_{\delta_0}.$$

Proof. In view of equation (3) and the affirmation of Lemma 1 we get

$$\varphi'(x) = \frac{\tilde{\omega}'(x)}{\tilde{\alpha}'[\varphi(x)]} = \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)} \cdot \frac{\tilde{\alpha}[\varphi(x)]}{\tilde{\alpha}'[\varphi(x)]} \le M \cdot \frac{\tilde{\omega}'(x)}{\tilde{\omega}(x)}, \quad x \in I_{\delta_0}.$$

Lemma 3. *Let the assumptions* (C2)–(C4) *be valid. Then the implicit equations*

$$\alpha_i(z_i) = \Omega_i(x), \ \ i = 1, 2, \dots, n \tag{4}$$

define on an interval $I_{\delta_2} \subset I_{x_0}$ *implicit functions*

$$z_i = \varphi_i(x) \equiv \alpha_i^{-1}[\Omega_i(x)], \ \ i = 1, 2, \dots, n \tag{5}$$

satisfying the properties

$$\varphi_i \in C^1(I_{\delta_2}, \mathbb{R}),\ \ \varphi_i \in I_{y_0} \quad \textit{and} \quad \varphi_i' > 0 \quad \textit{on} \quad I_{\delta_2},\ i = 1, 2, \dots, n; \tag{6}$$

$$\varphi_i(0^+) = 0, \quad i = 1, 2, \dots, n;$$

$$\varphi_i'(x) \equiv \frac{\Omega_i'(x)}{\alpha_i'[\varphi_i(x)]}\,, \ \ i = 1, 2, \dots, n. \tag{7}$$

If, moreover, there exists a constant $M \in \mathbb{R}^+$ such that

$$\alpha_i(y_i) \le M\alpha_i'(y_i), \;\; y \in I_{y_0}, \;\; i = 1, 2, \dots, n,$$

then

$$\varphi_i'(x) \le M \cdot \frac{\Omega_i'(x)}{\Omega_i(x)}, \quad i = 1, 2, \dots, n. \tag{8}$$

Proof. The proof follows immediately from Lemmas 1 and 2 if in their formulations $\tilde{\alpha} \equiv \alpha_i$ and $\tilde{\omega} \equiv \Omega_i$, $i = 1, 2, \dots, n$ are put since all Ω_i, $i = 1, 2, \dots, n$, defined by (2) satisfy necessary conditions. The value of δ_2 can be taken as minimal value of all corresponding δ_{0i}, $i = 1, 2, \dots, n$.

3. Singular problem (S$_+$), (IP)

Let us consider the singular problem (S$_+$) and (IP), i.e. the problem

$$g_i(x)\, y_i' = \sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x), \;\; i = 1, 2, \dots, n, \tag{S$_+$}$$

$$y_i(0^+) = 0, \;\; i = 1, 2, \dots, n. \tag{IP}$$

Theorem 1. *Suppose that conditions* (C1)–(C4) *are satisfied, there exist constants $k \in \mathbb{R}^+$, $M \in \mathbb{R}^+$, $k > 1$ and an interval $I_{x^{**}}$ with $x^{**} \le \min\left\{x_0 k^{-1}, y_0\right\}$ such that for $x \in I_{x^{**}}$:*

$$\omega_i(kx) > \omega_i(x) + kMg_i(x) \cdot \frac{\sum\limits_{j=1}^{n} C_{ji}\, \omega_j'(kx)}{\sum\limits_{j=1}^{n} C_{ji}\, \omega_j(kx)}, \quad i = 1, 2, \dots, n; \tag{i}$$

$$\alpha_i(x) \le M\alpha_i'(x), \quad i = 1, 2, \dots, n. \tag{ii}$$

Then there exist infinitely many solutions of the problem (S$_+$), (IP) *on an interval $I_{x^*} \subseteq I_{x^{**}}$.*

Proof. Let $\varphi_i(x)$, $i = 1, 2, \dots, n$ be the implicit functions defined on the interval I_{δ_2} by means of relations (4) or (5) (see Lemma 3). Let us define a domain Ω_1^0 of the form

$$\Omega_1^0 = \{(x, y) \in Q : x \in (0, \delta_3), \varphi_i(x) < y_i < \varphi_i(kx),\, i = 1, 2, \dots, n\},$$

supposing, without loss of generality, that $\delta_3 \le \delta_2$ is so small that $\varphi_i(kx) < y_0$, $i = 1, 2, \dots, n$ on I_{δ_3}. (Note that in accordance with Remark 1, $\varphi_i(x) < \varphi_i(kx)$, $i = 1, 2, \dots, n$, $x \in I_{\delta_3}$.)

Let us define auxiliary functions

$$u_i(x,y) \equiv u_i(x,y_i) \equiv \big(y_i - \varphi_i(x)\big)\big(y_i - \varphi_i(kx)\big), \ \ i = 1,2,\dots,n$$

and

$$v(x,y) \equiv v(x) \equiv x - \delta_3.$$

Then the domain Ω_1^0 can be written as

$$\Omega_1^0 = \{(x,y) \in Q : u_i(x,y) < 0, \ i = 1,2,\dots,n, \ v(x,y) < 0\}.$$

In the next we will show that all points of the sets

$$U_{1i} = \{(x,y) \in Q : u_i(x,y) = 0, u_j(x,y) \le 0, \ j = 1,2,\dots,n, \ j \ne i, \\ v(x,y) \le 0\}, \quad i = 1,2,\dots,n,$$

are the points of strict egress of the set Ω_1^0 with respect to the system (S_+). (For the corresponding definitions of this notion and for further details here and in the sequel we refer, e.g., to the book [9]. The notation used in the proof is taken from this book as well. Except this, the technique used is punctually explained e.g. in the papers [3,6,7] and [15].)

For verifying this we will compute the full derivatives of the functions $u_i(x,y)$, $i = 1,2,\dots,n$ along the trajectories of the system (S_+) on corresponding sets U_{1i}, $i = 1,2,\dots,n$. Let the index i be fixed. Then

$$\begin{aligned}\frac{du_i(x,y)}{dx} &= (y_i' - \varphi_i'(x))\,(y_i - \varphi_i(kx)) + \big(y_i - \varphi_i(x)\big)\big(y_i' - k\varphi_i'(kx)\big) = \\ &= \left[\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right]\big(y_i - \varphi_i(kx)\big) + \\ &+ \big(y_i - \varphi_i(x)\big)\left[\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - k\varphi_i'(kx)\right].\end{aligned}$$

If $(x,y) \in U_{1i}$ then either $y_i = \varphi_i(x)$ and $\varphi_j(x) \le y_j \le \varphi_j(kx)$, $j = 1,2,\dots,n$, $j \ne i$, or $y_i = \varphi_i(kx)$ and $\varphi_j(x) \le y_j \le \varphi_j(kx)$, $j = 1,2,\dots,n$, $j \ne i$. In the first case i.e. if

$$(x,y) \in U_{1i},\ y_i = \varphi_i(x), \ \varphi_j(x) \le y_j \le \varphi_j(kx),\ j = 1,2,\dots,n,\ j \ne i \tag{9}$$

we have

$$\begin{aligned}&\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{1i}, y_i = \varphi_i(x)} = \\ &= \left[\frac{1}{g_i(x)} \cdot \left(a_{ii}\alpha_i(\varphi_i(x)) + \sum_{j=1, j\ne i}^{n} a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right] \times \\ &\times \big(\varphi_i(x) - \varphi_i(kx)\big) = [\text{see (1) with } z_i = \varphi_i(x)] =\end{aligned}$$

$$\left[\frac{1}{g_i(x)}\cdot\left(\omega_i(x)-\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(\varphi_j(x))+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-\varphi_i'(x)\right]\times$$
$$\times\big(\varphi_i(x)-\varphi_i(kx)\big)=$$
$$=\left[-\frac{1}{g_i(x)}\cdot\sum_{j=1,j\neq i}^{n}a_{ij}\big[\alpha_j(\varphi_j(x))-\alpha_j(y_j)\big]-\varphi_i'(x)\right]\cdot\big(\varphi_i(x)-\varphi_i(kx)\big)\geq$$
$$\geq[\text{due to (C2)},(6)\text{ and }(9)]\geq-\varphi_i'(x)\big(\varphi_i(x)-\varphi_i(kx)\big)>0.$$

Thus, points $(x,y)\in U_{1i}$ if $y_i=\varphi_i(x)$ are points of strict egress. In the second case, i.e. if

$$(10)\qquad (x,y)\in U_{1i},\ y_i=\varphi_i(kx),\ \varphi_j(x)\leq y_j\leq\varphi_j(kx),\ j=1,2,\dots,n,\ j\neq i,$$

direct computation yields:

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{1i},y_i=\varphi_i(kx)}=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$
$$\times\left[\frac{1}{g_i(x)}\cdot\left(a_{ii}\alpha_i(\varphi_i(kx))+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-k\varphi_i'(kx)\right]=$$
$$=[\text{in view of (1) with }z_i=\varphi_i(x),(2),(4)\text{ and }(7)]=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$
$$\times\left[\frac{1}{g_i(x)}\cdot\left(\omega_i(kx)-\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j\big(\varphi_j(kx)\big)+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-\right.$$
$$\left.-k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right]=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$
$$\times\left[\frac{1}{g_i(x)}\cdot\left([\omega_i(kx)-\omega_i(x)]-\sum_{j=1,j\neq i}^{n}a_{ij}\big[\alpha_j(\varphi_j(kx))-\alpha_j(y_j)\big]\right)-\right.$$
$$\left.-k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right]\geq[\text{in view of (C2)},(2),(6),(8),(10)\text{ and (ii)}]\geq$$
$$\geq\big(\varphi_i(kx)-\varphi_i(x)\big)\left[\frac{\omega_i(kx)-\omega_i(x)}{g_i(x)}-kM\cdot\frac{\sum_{j=1}^{n}C_{ji}\omega_j'(kx)}{\sum_{j=1}^{n}C_{ji}\omega_j(kx)}\right]>$$
$$>[\text{in view of (i)}]>0.$$

This means that in both of the cases considered,

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{1i}} > 0, \quad i = 1,2,\dots,n. \tag{11}$$

So, points of the sets U_{1i}, $i = 1,2,\dots,n$ are the points of strict egress. Inequality (11) simultaneously says that, if orientation of the x-axis is changed into reverse orientation, points $(x,y) \in U_{1i}$, $i = 1,2,\dots,n$ are points of strict ingress and every point of the set

$$S = \{(x,y) \in Q : x = \delta_3,\ \varphi_i(x) < y_i < \varphi_i(kx),\ i = 1,2,\dots,n\}$$

defines a unique solution $y = y^*(x)$ such that $(x, y^*(x)) \in \Omega_1^0$ on interval I_{δ_3}, i.e. this solution solves the problem (S_+), (IP). Put $x^* = \delta_3$. Now Theorem 1 is proved.

Corollary 1. *The affirmation of the Theorem* 1 *can be improved. Namely, as it follows from proof above, there exist infinitely many solutions $y = y^*(x)$ of the problem* (S_+), (IP)*, each of which satisfies, on the interval I_{x^*}, the inequalities*

$$\varphi_i(x) < y_i^*(x) < \varphi_i(kx),\ \ i = 1,2,\dots,n.$$

4. Singular problem (S_-), (IP)

Now consider the singular problem (S_-), (IP), i.e. the problem

$$g_i(x)\, y_i' = -\sum_{j=1}^{n} a_{ij}\alpha_j(y_j) + \omega_i(x),\ \ i = 1,2,\dots,n, \tag{S_-}$$

$$y_i(0^+) = 0,\ \ i = 1,2,\dots,n. \tag{IP}$$

Theorem 2. *Suppose that conditions* (C1)–(C4) *are satisfied, there exist constants $k \in \mathbb{R}^+$, $M \in \mathbb{R}^+$, $k < 1$ and an interval $I_{x^{**}}$ with $x^{**} \le \min\{x_0, y_0\}$ such that for $x \in I_{x^{**}}$:*

$$\omega_i(x) > \omega_i(kx) + kMg_i(x)\cdot \frac{\sum\limits_{j=1}^{n} C_{ji}\,\omega_j'(kx)}{\sum\limits_{j=1}^{n} C_{ji}\,\omega_j(kx)}\,, \quad i = 1,2,\dots,n \tag{iii}$$

and the condition (ii) *holds. Then there exists at least one solution $y = y^*(x)$ of the problem* (S_-), (IP) *on an interval $I_{x^*} \subseteq I_{x^{**}}$.*

Proof. Introduce a domain Ω_2^0 of the form

$$\Omega_2^0 = \{(x,y) \in Q : x \in (0,\delta_2), \varphi_i(kx) < y_i < \varphi_i(x),\ i = 1,2,\dots,n\},$$

where δ_2 was defined in Lemma 3 and $\varphi_i(x)$, $i = 1, 2, \ldots, n$ are defined as in the proof of Theorem 1. (Note that in the case considered $\varphi_i(kx) < \varphi_i(x)$, $i = 1, 2, \ldots, n$, $x \in I_{\delta_2}$.)

Let us define auxiliary functions

$$u_i(x, y) \equiv u_i(x, y_i) \equiv \big(y_i - \varphi_i(x)\big)\big(y_i - \varphi_i(kx)\big), \ \ i = 1, 2, \ldots, n$$

and

$$v(x, y) \equiv v(x) \equiv x - \delta_2.$$

Then the domain Ω_2^0 can be written as

$$\Omega_2^0 = \{(x, y) \in Q : u_i(x, y) < 0, \ i = 1, 2, \ldots, n, \, v(x, y) < 0\}.$$

In the following we will show that all points of the sets

$$\begin{gathered} U_{2i} = \{(x, y) \in Q : u_i(x, y) = 0, u_j(x, y) \le 0, \ j = 1, 2, \ldots, n, \, j \ne i, \\ v(x, y) \le 0\}, \quad i = 1, 2, \ldots, n, \end{gathered}$$

are the points of strict ingress of the set Ω_2^0 with respect to the system (S_-).

Analogously as in the proof of Theorem 1 we compute the full derivatives of the functions $u_i(x, y)$, $i = 1, 2, \ldots, n$ along the trajectories of the system (S_-) on corresponding sets U_{2i}, $i = 1, 2, \ldots, n$. Let the index i be fixed. Then

$$\begin{aligned} \frac{du_i(x, y)}{dx} &= (y_i' - \varphi_i'(x))\,(y_i - \varphi_i(kx)) + \big(y_i - \varphi_i(x)\big)\big(y_i' - k\varphi_i'(kx)\big) = \\ &= \left[-\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^n a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right] \big(y_i - \varphi_i(kx)\big) + \\ &\quad + \big(y_i - \varphi_i(x)\big) \left[-\frac{1}{g_i(x)} \cdot \left(\sum_{j=1}^n a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - k\varphi_i'(kx)\right]. \end{aligned}$$

If $(x, y) \in U_{2i}$ then either $y_i = \varphi_i(x)$ and $\varphi_j(kx) \le y_j \le \varphi_j(x)$, $j = 1, 2, \ldots, n$, $j \ne i$, or $y_i = \varphi_i(kx)$ and $\varphi_j(kx) \le y_j \le \varphi_j(x)$, $j = 1, 2, \ldots, n$, $j \ne i$. In the first case i.e. if

$$(x, y) \in U_{2i}, \, y_i = \varphi_i(x), \, \varphi_j(kx) \le y_j \le \varphi_j(x), \, j = 1, 2, \ldots, n, \, j \ne i \tag{12}$$

we have

$$\begin{gathered} \left.\frac{du_i(x, y)}{dx}\right|_{(x,y) \in U_{2i}, y_i = \varphi_i(x)} = \\ = \left[-\frac{1}{g_i(x)} \cdot \left(a_{ii}\alpha_i(\varphi_i(x)) + \sum_{j=1, j\ne i}^n a_{ij}\alpha_j(y_j) - \omega_i(x)\right) - \varphi_i'(x)\right] \times \\ \times \big(\varphi_i(x) - \varphi_i(kx)\big) = [\text{see (1) with } z_i = \varphi_i(x)] = \end{gathered}$$

$$\left[\frac{-1}{g_i(x)}\cdot\left(\omega_i(x)-\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(\varphi_j(x))+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-\varphi_i'(x)\right]\times$$

$$\times\big(\varphi_i(x)-\varphi_i(kx)\big)=$$

$$=\left[\frac{1}{g_i(x)}\cdot\sum_{j=1,j\neq i}^{n}a_{ij}\left[\alpha_j(\varphi_j(x))-\alpha_j(y_j)\right]-\varphi_i'(x)\right]\cdot\big(\varphi_i(x)-\varphi_i(kx)\big)\le$$

$$\le[\text{due to (C2)}, (6) \text{ and } (12)]\le-\varphi_i'(x)\big(\varphi_i(x)-\varphi_i(kx)\big)<0.$$

Thus, points $(x,y)\in U_{2i}$ if $y_i=\varphi_i(x)$ are points of strict ingress. In the second case, i.e. if

(13) $\qquad (x,y)\in U_{2i},\ y_i=\varphi_i(kx),\ \varphi_j(kx)\le y_j\le\varphi_j(x),\ j=1,2,\dots,n,\ j\neq i,$

direct computation yields:

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{2i},y_i=\varphi_i(kx)}=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$

$$\times\left[-\frac{1}{g_i(x)}\cdot\left(a_{ii}\alpha_i(\varphi_i(kx))+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-k\varphi_i'(kx)\right]=$$

$$=[\text{in view of (1) with } z_i=\varphi_i(x), (2), (4) \text{ and } (7)]=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$

$$\times\left[-\frac{1}{g_i(x)}\cdot\left(\omega_i(kx)-\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j\big(\varphi_j(kx)\big)+\sum_{j=1,j\neq i}^{n}a_{ij}\alpha_j(y_j)-\omega_i(x)\right)-\right.$$

$$\left.-k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right]=\big(\varphi_i(kx)-\varphi_i(x)\big)\times$$

$$\times\left[\frac{1}{g_i(x)}\cdot\left([\omega_i(x)-\omega_i(kx)]+\sum_{j=1,j\neq i}^{n}a_{ij}\left[\alpha_j(\varphi_j(kx)-\alpha_j(y_j)\right]\right)-\right.$$

$$\left.-k\frac{\Omega_i'(kx)\alpha_i\big(\varphi_i(kx)\big)}{\Omega_i(kx)\alpha_i'\big(\varphi_i(kx)\big)}\right]\le[\text{in view of (C2)}, (2), (6), (8), (13) \text{ and (ii)}]\le$$

$$\le\big(\varphi_i(kx)-\varphi_i(x)\big)\left[\frac{\omega_i(x)-\omega_i(kx)}{g_i(x)}-kM\cdot\frac{\sum_{j=1}^{n}C_{ji}\omega_j'(kx)}{\sum_{j=1}^{n}C_{ji}\omega_j(kx)}\right]<$$

$$<[\text{in view of (iii)}]<0.$$

This means that in both of the cases considered

$$\left.\frac{du_i(x,y)}{dx}\right|_{(x,y)\in U_{2i}} < 0, \quad i = 1,2,\dots,n. \tag{14}$$

So, points of the sets U_{2i}, $i = 1,2,\dots,n$ are the points of strict ingress. Inequality (14) simultaneously says that, if orientation of the x-axis is changed into reverse orientation, points $(x,y) \in U_{2i}$, $i = 1,2,\dots,n$ are points of strict egress. Let us define the set

$$S = \{(x,y) \in Q : x = \delta_2,\ \varphi_i(kx) \le y_i \le \varphi_i(x),\ i = 1,2,\dots,n\}.$$

It is easy to show that its boundary

$$\partial S = \left\{(x,y) \in \bigcup_{i=1}^{n} U_{2i} : x = \delta_2\right\}$$

is not a retract of S but is a retract of the set $\bigcup\limits_{i=1}^{n} U_{2i}$. Then, according to Ważewski's principle, there is a point $(\delta_2, y^*) \in S \setminus \partial S$ such that the graph of corresponding solution $y = y^*(x)$ with $y^*(\delta_2) = y^*$ lies in the domain Ω_2^0 for $x \in (0,\delta_2]$. Therefore this solution solves simultaneously the problem (S_-), (IP). Put $x^* = \delta_2$. The theorem is proved.

Corollary 2. *The affirmation of the Theorem* 2 *can be improved. The solution* $y = y^*(x)$ *of the problem* (S_-), (IP) *as it follows from the proof of Theorem* 2, *satisfies the inequalities*

$$\varphi_i(kx) < y_i^*(x) < \varphi_i(x),\ \ i = 1,2,\dots,n,\ x \in (0,x^*].$$

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