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MULTISTRUCTURES DETERMINED BY DIFFERENTIAL RINGS

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Abstract. Multistructures namely hypergroups are playing very essential role in contemporary mathematics. This contribution aims at some natural constructions of such multistructures defined on differential rings with differentiation operators which can be especially applied to rings of continuously differentiable functions.

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The systematic study of algebraic aspects of transformations of differential and difference operators applied to investigation of differential and difference equations is of a persistent interest. General algebraic approach to the transformation theory is described in [13], more in detail see also [12] and other related papers of Professor Neuman. This fruitful direction has been iniciated by Professor O. Borůvka in the 1950s in the framework of his intensive research of linear differential transformations of the second order - [3]. The theory dominating by high level of algebraization and geometrization is developing by the Borůvka's school and his succesors up to present times.

In contemporary investigations of algebraic and geometrical structures an important role is playing by hyperstructures, formerly called multistructures, which occur very naturally in convexity theory, harmonic analysis, in projective and affine geometry, in the decomposition theory of noncommutative algebraic structures and elsewhere, cf. [2,4,6,7,8,11,14,15,16].

In this contribution we give construction of multistructures determined by quasi-orders defined by means of derivation operators on differential rings. Some constructions based on results of [4], par.1 chapter IV and of paper [5] are possible for general differential rings, the other are specialized.

Recall basic concepts overtaken e.g. from [6]. A *multigroupoid* or a *hypergroupoid* (in recent literature) is a pair $(M, \cdot)$, where M is a nonempty set and a mapping $\cdot : M \times M \to \mathcal{P}^*(M)$ (the system of all nonempty subsets of M) is a binary *multioperation* called also a *hyperoperation*. This multioperation is usually extended onto the powerset $\mathcal{P}(M)$ by the rule $A.B = \bigcup\{a.b; a \in A, b \in B\}$ for any pair $A \neq \emptyset \neq B$, where $A, B \subset M$ and moreover by $\emptyset.A = \emptyset = A.\emptyset$. It is to be noted that operations on powersets of carriers of ternary relational structures were used by Professor M. Novotný in a series of his papers - started by [14] - including also investigations of relationships between ternary structures and multistructures. If this multioperation is associative (here $A.B = \bigcup\{a.b; a \in A, b \in B\}$ for any pair $A \neq \emptyset \neq B$, $A, B \subseteq M$) then $(M, \cdot)$ is called a *semihypergroup*, if $(M, \cdot)$, moreover, satisfies the reproduction axiom - $a.M = M = M.a$ for any $a \in M$ - then $(M, \cdot)$ is said to be a *multigroup* or a *hypergroup*. We will use the latter terms. A hypergroupoid satisfying the reproduction axiom is called a *quasi-hypergroup*.

Let $(R, +, ., \Delta_R)$ be a commutative *differential ring*, i.e. $(R, +, .)$ is a commutative ring, Δ_R is a set of derivations on the set R, which means that Δ_R is a subset of the endomorphism monoid $\mathrm{End}(R, +)$ of the additive abelian group of the ring $(R, +, .)$ satisfying the differentiation rule. Thus for $d \in \Delta_R$ and any pair of elements $x, y \in R$ we have $d(x + y) = d(x) + d(y)$ and $d(x.y) = d(x).y + x.d(y)$. Moreover we suppose that any $d : R \to R$ is surjective. A differential structure Δ_R of a ring can be endowed with the Lie multiplication $d_1 \diamond_L d_2 = d_1 d_2 - d_2 d_1$; then $(R, +, \diamond_L)$ is a Lie ring of derivations. If $\Delta_R = \{d\}$ is a singleton we say that this differential structure is *monogeneous*.

By $\mathbf{R}, \mathbf{R}^+, \mathbf{N}$ we denote the set of all real, positive real numbers, positive integers, respectively.

Examples 1. Let $J = (a, b) \subseteq \mathbf{R}$ (possibly $J = \mathbf{R}$) and $\mathbf{C}^\infty(J)$ - as usually - be the ring of real functions $f : J \to \mathbf{R}$ with continuous derivatives of all orders. If $\Delta = \{\frac{d}{dx}\}$, where $\frac{df}{dx} = f'$ is the usual derivative of a function $f \in \mathbf{C}^\infty(J)$, then $(\mathbf{C}^\infty(J), +, ., \Delta)$ is a differential ring with a monogenous differential structure.

2. Let $\mathbf{R}[x_1, \ldots, x_n]$, $[x_1, \ldots, x_n] \in \mathbf{R}^n$ (for a fixed integer n) be the ring of all polynomials with coefficients in the field $(\mathbf{R}, +, .)$. Denoting $\Delta = \{\sum_1^n \lambda_k . \frac{\partial}{\partial x_k}; [\lambda_1, \ldots, \lambda_n] \in \mathbf{R}^n\}$ we obtain $(\mathbf{R}[x_1, \ldots, x_n], +, ., \Delta)$ as an example of a differential ring.

Other examples can be found e.g. in [9,10] The join operation $\cdot$ in a hypergroupoid $(M, \cdot)$ has two inverses - right extension and left extension - defined by $a/b = \{x; a \in x \cdot b\}$ and $b\backslash a = \{x; a \in b \cdot x\}$ called also *right* and *left fractions*, respectively. The reproductive axiom for $(M, \cdot)$ is easily seen to be equivalent to the condition that fractions $a/b, b\backslash a$ are nonempty for any pair $a, b \in M$. In the case of a commutative join operation $\cdot$ evidently $a/b = b\backslash a$. Now, a hypergroup $(M, \cdot)$ is called a *join space* if it is commutative and satisfies the *transposition axiom*: For any quadruple $a, b, c, d \in M$ the implication $a/b \cap c/d \neq \emptyset \Rightarrow a.d \cap c.d \neq \emptyset$ is valid -

[6,7,8]. The concept of a join space has been introduced by W. Prenowitz and used by him and afterwards by him and J. Jantosciak to build again several branches of geometry. Recall that a self-map f of a hypergroupoid $(M, .)$ is called a *good endomorphism* of $(M, .)$ if it satisfies these set equalities $f(x, y) = f(x).f(y)$ for any pair $x, y \in M$.

Let $(R, +, \cdot, \Delta_R)$ be a differential (non necessary commutative) ring, $M(\Delta_R)$ be the free monoid over Δ_R within the full transformation monoid of R (i.e. $M(\Delta_R)$ is the set of all finite words $d_1 \dots d_n$, $d_k \in \Delta_R$ including the empty word Λ, identified with the identity operator id_R, endowed with the binary operation of concatenation). We define $d_1 \dots d_n(x) = d_n(\dots(d_1(x))\dots)$ which means application of the composition of operators $d_1, \dots, d_n$- in this order - to the element $x \in R$.

Theorem 1. *Let $(R, +, \cdot, \Delta_R)$ be a differential ring. Let $x * y = \{d_1 \dots d_n(z); z \in \{x, y\}, d_k \in \Delta_R, n \in \mathbf{N}\} = \{\delta(z); z \in \{x, y\}, \delta \in M(\Delta_R)\}$. Then we have*

$1°$ *$(R, *)$ is a commutative hypergroup such that any differential endomorphism of the ring $(R, +, \cdot, \Delta_R)$ (i.e. $f \in \mathrm{End}(R, +, \cdot)$ with $f(d_k(x)) = d_k(f(x)), x \in R$) is a good endomorphism of $(R, *)$.*

$2°$ *The hypergroup $(R, *)$ satisfies the transposition law, hence it is a join space if and only if for any pair of elements $x, y \in R$ such that there exists a pair of words $(\delta, \sigma) \in M(\Delta_R) \times M(\Delta_R)$ and a suitable element $z \in R$ with $\delta(z) = x, \sigma(z) = y$, we have $\tau(x) = \omega(y)$ for some pair of words $\tau \in M(\Delta_R)$, $\omega \in M(\Delta_R)$.*

Proof. Define a binary relation $r \subset R \times R$ by xry whenever there exists an m-tuple of derivations operators $d_1, \dots, d_m \in \Delta_R$, i.e. a word $\delta = d_1 \dots d_m \in M(\Delta_R)$ such that $y = \delta(x)$. The relation r is reflexive (if $d_1 = \dots = d_m = id_R$) and transitive: For $x, y, z \in R$ such that xry, yrz, i.e. $y = \delta(x), z = \sigma(y)$ for suitable words $\delta, \sigma \in M(\Delta_R)$ we get $z = \delta\sigma(x) = \sigma(\delta(x))$, with $\delta\sigma \in M(\Delta_R)$, thus xrz. If for arbitrary pair $x, y \in R$ we define

$$x * y = \{\delta(z); z \in \{x, y\}, \delta \in M(\Delta_R)\} =$$
$$= \{\delta(x); \delta \in M(\Delta_R)\} \cup \{\delta(y); \delta \in M(\Delta_R)\} = r(x) \cup r(y)$$

then by the fundamental construction [4], or [5] and [16] we have that $(R, *)$ is a commutative hypergroup. Further, if $f : R \to R$ is a differential endomorphism of the ring $(R, +, ., \Delta_R)$ which means $f \in \mathrm{End}(R, +, .)$ and $f(d(x)) = d(f(x))$ for any $d \in \Delta_R$ and any $x \in R$, then by the induction $f(\delta(x)) = \delta(f(x))$ for any word $\delta \in M(\Delta_R)$ and each element $x \in R$, thus for any pair $x, y \in R$ we have

$$f(x * y) = \{f(\delta(z)); z \in \{x, y\}, \delta \in M(\Delta_R)\} =$$
$$= \{\delta(f(z)); z \in \{x, y\}, \delta \in M(\Delta_R)\} = f(x) * f(y).$$

Hence the assertion $1°$ is true.

Finally, the monoid $M(\Delta_R)$ acts on the set R. By [5] Theorem 6 the hypergroup $(R, *)$ is a join space iff for every pair of elements $x, y \in R$ such that there exists a pair of words $\delta_1, \sigma_1 \in M(\Delta_R)$ and an element $z \in R$ with $\delta(z) = x, \sigma(z) = y$, we have $\tau(x) = \omega(y)$ for suitable words $\tau, \omega \in M(\Delta_R)$, thus we obtain the assertion $2°$.

Remark. Using principal ideals within differential images of the carrier set R of a differential ring $(R, +, ., \{d\})$ with a monogeneous differential structure, we can construct a countable set (in general) of commutative extensive hypergroups $(R, \circ_m)$ with the same carrier R. (Extensivity of a hyperoperation $\circ$ means $x, y \in x \circ y$ for all $x, y \in R$.) This construction is based on [4], chapt.IV, Theorem 2.1 which is generalized in [15] - Propositions 2,3. More in detail, for a given positive integer $m \in \mathbf{N}$ we define

$$x \circ_m y = \{z \in R; x.d^m(R) \subseteq z.d^m(R) \ \textit{or}\ y.d^m(R) \subseteq z.d^m(R)\},$$

where $d^m(R) = \{d^m(x); x \in R\}$. Then by the above mentioned theorems we obtain that $(R, \circ_m)$ is a commutative extensive hypergroup.

Theorem 2. *Let $(R, +, \cdot, \Delta_R)$ be a commutative differential ring with a monogeneous differential structure $\Delta_R = \{d\}$. Let $(R, *_d)$ be a commutative hypergroupoid defined by the indefinite integral $x *_d y = d^{-1}(x + y)$ for all $x, y \in R$. Then $(R, *_d)$ is a commutative quasi-hypergroup such that $(x + y)/(u + v) = x/u + y/v$ for any quadruple $x, y, u, v \in R$ and for arbitrary triad $x, y, z \in R$ we have*

1° $x/y = d(x) - y$,

2° $d(x) = (x + y)/z - y/z$,

3° $d(x/y) = d(x)/d(y)$,

4° $d(x *_d x + y *_d y) = d(x *_d y) + d(x *_d y)$.

Proof. We show first that the hypergroupoid $(R, *_d)$ satisfies the reproduction axiom.

Let $a \in R$ be an arbitrary element. Since $a *_d R \subseteq R$ and $(R, *_d)$ is commutative if suffices to prove the inclusion $R \subseteq a *_d R$. For any $x \in R$ then $d^{-1}(x) = I(x) = \{y \in R; d(y) = x\}$ is called the indefinite integral of x. Now, for arbitrary $b \in R$ we denote $x_b = d(b) - a$. Then $d(b) = a + x_b$, i.e. $b \in d^{-1}(a + x_b) = I(a + x_b) = a *_d x_b \subseteq \bigcup_{x \in R} a *_d x = a *_d R$, hence $a *_d R = R = R *_d a$ for any $a \in R$. It is easy to see that $(R, *_d)$ is not associative in general, thus $(R, *_d)$ is a commutative quasi-hypergroup. Further, for $x, y, u, v \in R$ arbitrary we have

1° $x/y = \{z \in R; x \in z *_d y\} = \{z \in R; x \in I(z + y)\}$, thus $x \in z *_d y$ iff $d(x) = z + y$, thus $z = d(x) - y$, hence we get that $z \in x/y$ iff $z = d(x) - y$ consequently $x/y = d(x) - y$ which is a singleton.
Now $x/u + y/v = d(x) - u + d(y) - v = d(x + y) - (u + v) = (x + y)/(u + v)$.

2° For any $x, y, z \in R$ we have $(x + y)/z = d(x + y) - z = d(x) + d(y) - z = d(x) + y/z$, therefore $d(x) = (x + y)/z - y/z$. Similarly,

3° $d(x/y) = d(d(x) - y) = d(d(x)) - d(y) = d(x)/d(y)$ and

4° $d(x *_d y) + d(x *_d y) = d(d^{-1}(x + y)) + d(d^{-1}(x + y)) = x + y + x + y = x+x+y+y = d(d^{-1}(x+x+y+y)) = d(d^{-1}(x+x)+d^{-1}(y+y)) = d^{-1}(x*_dx+y*_dy)$.

Now we specialize our considerations to the classical differential rings of real functions $f \in \mathbf{C}^\infty(J)$, $J = (a, b) \subseteq \mathbf{R}$ (not excluding the case $J = \mathbf{R}$) with the usual differentiation. For any $f \in \mathbf{C}^\infty(J)$ we denote by $\int f(x)dx$ the set of all primitive functions to f, i.e.$\int f(x)dx = \{F : J \to \mathbf{R}; F'(x) = f(x), x \in J\}$. For any pair of function $\varphi, \psi \in \mathbf{C}^\infty(J)$ we define a hyperoperation $\star$ on the ring $\mathbf{C}^\infty(J)$ by

$$f \star_{(\varphi,\psi)} g = \int (\varphi'(x)f(x) + \psi'(x)g(x))dx, \quad f, g \in \mathbf{C}^\infty(J).$$

Evidently, $(\mathbf{C}^\infty(J), \star_{(\varphi,\psi)})$ is a hypergroupoid (noncommutative in general).

Theorem 3. *Let $J \subseteq \mathbf{R}$ be an open interval, $\varphi, \psi \in \mathbf{C}^\infty(J)$ be a pair of strictly monotone functions (i.e. $\varphi'(x).\psi'(x) \neq 0$ for all $x \in J$). Then the hypergroupoid $(\mathbf{C}^\infty(J), \star_{(\varphi,\psi)})$ is a quasi-hypergroup (i.e. it satisfies the reproduction axiom) which is commutative if and only if the difference $\varphi - \psi$ on the interval J is a constant function.*

Proof. Clearly, for any pair $f, g \in \mathbf{C}^\infty(J)$ and any function $h \in f \star_{(\varphi_1,\varphi_2)} g$ we have $h \in \mathbf{C}^\infty(J)$. Suppose $f \in \mathbf{C}^\infty(J)$ is an arbitrary function. Then evidently

$$f \star_{(\varphi_1,\varphi_2)} \mathbf{C}^\infty(J) = \bigcup\{f \star_{(\varphi_1,\varphi_2)} g; g \in \mathbf{C}^\infty(J)\} \subseteq \mathbf{C}^\infty(J)$$

and
$\mathbf{C}^\infty(J) \star_{(\varphi_1,\varphi_2)} f \subseteq \mathbf{C}^\infty(J)$, as well. We prove the opposite inclusions.
Suppose that $g \in \mathbf{C}^\infty(J)$ is an arbitrary function. Define

$$h_1(x) = \tfrac{1}{\varphi_2'(x)}(g'(x) - \varphi_1'(x)f(x)), \quad x \in J.$$

Since $\varphi_1'(x).\varphi_2'(x) \neq 0$ for each $x \in J$, then $\varphi_2'(x) \neq 0$ for any $x \in J$, thus the function $\frac{1}{\varphi_2'(x)}$ is defined on the interval J and $\frac{1}{\varphi_2'(x)} \in \mathbf{C}^\infty(J)$, $g'(x) - \varphi_1'(x)f(x) \in \mathbf{C}^\infty(J)$, hence $h_1 \in \mathbf{C}^\infty(J)$. Then

$$f \star_{(\varphi_1,\varphi_2)} h_1 = \int(\varphi_1'(x)f(x) + \varphi_2'(x)h_1(x))dx = \int g'(x)dx = \{g(x) + c; c \in \mathbf{R}\},$$

thus

$$g \in f \star_{(\varphi_1,\varphi_2)} h_1 \subseteq \bigcup\{f \star_{(\varphi_1,\varphi_2)} h; h \in \mathbf{C}^\infty(J)\}.$$

Similarly if we define

$$h_2(x) = \tfrac{1}{\varphi_1'(x)}(g'(x) - \varphi_2'(x)f(x)), \quad x \in J,$$

then the assumption $\varphi_1'(x) \neq 0$ for any $x \in J$ and $f, g, \varphi_1, \varphi_2 \in \mathbf{C}^\infty(J)$ implies $h_2 \in \mathbf{C}^\infty(J)$. Further,

$$h_2 \star_{(\varphi_1,\varphi_2)} f = \int(\varphi_1'(x)h_2(x) + \varphi_2'(x)f(x))dx = \int g'(x)dx = \{g(x) + c; c \in \mathbf{R}\},$$

thus - similarly as above - we have

$$g \in h_2 \star_{(\varphi_1,\varphi_2)} f \subseteq \bigcup\{h \star_{(\varphi_1,\varphi_2)} f; h \in \mathbf{C}^\infty(J)\} = \mathbf{C}^\infty(J) \star_{(\varphi_1,\varphi_2)} f.$$

Hence

$$\mathbf{C}^\infty(J) \subseteq (f \star_{(\varphi_1,\varphi_2)} \mathbf{C}^\infty(J)) \cap (\mathbf{C}^\infty(J) \star_{(\varphi_1,\varphi_2)} f),$$

consequently the hypergroupoid $(\mathbf{C}^\infty(J), \star_{(\varphi_1,\varphi_2)})$ satisfies the reproduction axiom. Therefore it is a quasi-hypergroup.

Now suppose $\varphi_1(x) - \varphi_2(x) = c$ for some real number $c \in \mathbf{R}$. Then $\varphi_1' = \varphi_2'$ and $f \star_{(\varphi_1,\varphi_2)} g = g \star_{(\varphi_1,\varphi_2)} f$ for any pair of functions $f, g \in \mathbf{C}^\infty(J)$. On the contrary, if the hyperoperation $\star_{(\varphi_1,\varphi_2)}$ is commutative then $\int(\varphi_1'(x)f(x) + \varphi_2'(x)g(x))dx = \int(\varphi_1'(x)g(x) + \varphi_2'(x)f(x))dx$ which is equivalent to

$$\int(\varphi_1'(x) - \varphi_2'(x))(f(x) - g(x))dx = 0. \tag{1}$$

Especially for $f(x) = g(x) + 1, x \in J$ the equality (1) gives $\int(\varphi_1'(x) - \varphi_2'(x))dx = 0$, which implies $\varphi_1'(x) - \varphi_2'(x) = 0$ thus $\varphi_1(x) - \varphi_2(x)$ is a constant function.

Remark. It is easy to see that the hyperoperation

$$\star_{(\varphi_1,\varphi_2)} : \mathbf{C}^\infty(J) \times \mathbf{C}^\infty(J) \to \mathcal{P}^*(\mathbf{C}^\infty(J))$$

is not associative. In a special case $\varphi_1(x) = \varphi_2(x) = x, x \in J$, i.e. within the commutative quasi-hypergroup $(\mathbf{C}^\infty(J), *)$, where $f * g = \int(f(x) + g(x))dx$ for any pair $f, g \in \mathbf{C}^\infty(J)$, we get from Theorem 2 $(1^\circ, 2^\circ, 3^\circ)$ the following rules:

$$f(x)/g(x) = \frac{df(x)}{dx} - g(x),\ \frac{d}{dx}(f(x)/g(x)) = \frac{df(x)}{dx}/\frac{dg(x)}{dx},$$
$$\frac{df(x)}{dx} = (f(x) + g(x))/h(x) - g(x)/h(x)$$

for arbitrary $f, g, h \in \mathbf{C}^\infty(J)$. Moreover, for any quadruple $f, g, u, v \in \mathbf{C}^\infty(J)$ then we have $(f(x)+g(x))/(u(x)+v(x)) = f(x)/u(x)+g(x)/v(x)$. Using derivatives of functions from $\mathbf{C}^\infty(J)$ we can expressed certain sufficient conditions for validity of transposition law for the quasi-hypergroup $(\mathbf{C}^\infty(J), \star_{(\varphi_1,\varphi_2)})$. Moreover, transposition hypergroups, forming an important class of hypergroups, can be constructed from quasi-ordered groups and monoids of some transformation operators of rings of continuously differentiable functions. These operators yielding substitutions for some classes of ordinary differential equations will be investigated in a forthcoming paper.

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