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LINEAR DIFFERENTIAL EQUATIONS WITH SEVERAL UNBOUNDED DELAYS

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Abstract. The paper is concerned with the asymptotic estimate of the solutions of the delay differential equation

$$\dot{x}(t) = -a(t)x(t) + b_1(t)x(\tau_1(t)) + b_2(t)x(\tau_2(t))$$

with the continuous coefficients $a(t)$, $b_1(t)$, $b_2(t)$ and the unbounded lags. We derive the conditions under which each solution of this equation can be estimated in the terms of a solution of the system of Schröder's functional equations.

AMS Subject Classification. 34K25, 39B22

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1. Introduction

We study the functional differential equation

$$\dot{x}(t) = -a(t)x(t) + b_1(t)x(\tau_1(t)) + b_2(t)x(\tau_2(t)), \qquad t \in I = [t_0, \infty), \tag{1}$$

where $a(t)$ is a positive continuous function on I, $b_i(t)$ are continuous functions on I, $\tau_i(t)$ are continuously differentiable and unbounded functions on I such that $\tau_i(t_0) = t_0$, $\tau_i(t) < t$ for every $t > t_0$, $\dot{\tau}_i(t_0) < 1$ and $\dot{\tau}_i(t)$ are nonincreasing on I, $i = 1, 2$. We assume that all these conditions are fulfilled throughout the whole paper.

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The investigation of these equations has been motivated by the equation

$$\dot{x}(t) = a\,x(t) + b\,x(\lambda t), \qquad 0 < \lambda < 1 \tag{2}$$

arising in the problem of the motion of a pantograph head on an electric locomotive. Equation (2) and its modifications has been subject of numerous investigations (for the methods and results see, e.g., G. Derfel [4] , A. Iserles [6], T. Kato and J. B. McLeod [7], E. B. Lim [9], Y. Liu [10], G. Makay and J. Terjéki [11], L. Pandolfi [13] and papers [2], [3]). In this paper, we wish to extend some of the asymptotic results discussed in these papers to the case of the equation (1).

2. Preliminaries

Choose any $\sigma \in I$ and let $\sigma^* = \min\{\tau_i(\sigma),\, i = 1,2\}$. By a solution of (1) we understand a real valued function $x(t) \in C^0([\sigma^*,\infty)) \cap C^1([\sigma,\infty))$ such that $x(t)$ satisfies (1) for every $t \geq \sigma$.

The key tool in our investigations is the theory of functional equations in a single variable. The survey of the methods and results concerning this theory can be found in the book M. Kuczma, B. Choczewski, R. Ger [8]. In this section, we mention the problem of the existence of the simultaneous solution of the system of the Schröder's equations

$$\begin{aligned} \varphi(\tau_1(t)) &= \lambda_1 \varphi(t),\\ \varphi(\tau_2(t)) &= \lambda_2 \varphi(t), \end{aligned} \tag{3}$$

where $t \in I$, λ_1 and λ_2 are suitable reals parameters. We have the following

Proposition 1. *Let $\lambda_1 = \dot{\tau}_1(t_0)$, $\lambda_2 = \dot{\tau}_2(t_0)$ and $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$ on I. Then the system* (3) *has a solution $\varphi(t) \in C^1(I)$ with a positive and bounded derivative on I.*

Proof. First we consider a single equation of the system (3), e.g.,

$$\varphi(\tau_1(t)) = \lambda_1 \varphi(t), \qquad t \in I. \tag{4}$$

The existence of the solution $\varphi(t) \in C^1(I)$ having a positive derivative on I follows from the classical result of the theory of functional equations (see, e.g., [8]). Differentiating (4) we obtain

$$\dot{\varphi}(\tau_1(t)) = \frac{\lambda_1}{\dot{\tau}_1(t)} \dot{\varphi}(t).$$

The inequality $\lambda_1 \big/ \dot{\tau}_1(t) \geq 1$ now implies the boundedness of $\dot{\varphi}(t)$ on I.

It remains to show that $\varphi(t)$ defines also a solution of the latter equation of (3). This problem has been dealt with in [1] (see also F. Neuman [12] and M. Zdun [14]). By Proposition 3 of [1], the necessary and sufficient condition for the existence of the simultaneous solution $\varphi(t)$ of (3) is the commutativity of the couple $\tau_1(t)$, $\tau_2(t)$. □

Remark 1. The required solution of (3) can be given in several important cases explicitly. These cases are discussed in Section 4.

3. Asymptotic behaviour of the solutions

In this section, we mention the main result concerning equation (1).

Theorem 1. *Let $\tau_1(t)$, $\tau_2(t)$ be commutable functions on the interval I. Let $\lambda_1 = \dot{\tau}_1(t_0)$, $\lambda_2 = \dot{\tau}_2(t_0)$ and let $\varphi(t) \in C^1(I)$ be a solution of (3) with a positive and bounded derivative on I. Let $x(t)$ be a solution of (1), where $a(t) \geq K/(\varphi(t))^\beta$ and $0 < |b_1(t)| + |b_2(t)| \leq La(t)$ for every $t \in I$ and suitable reals $K, L > 0$, $\beta < 1$. Then*

$$x(t) = O((\varphi(t))^\alpha) \qquad \text{as } t \to \infty, \qquad \alpha = \frac{\log L}{\log \lambda^{-1}}, \quad \lambda = \max(\lambda_1, \lambda_2). \tag{5}$$

Proof. The function $\varphi(t)$ is obviously positive for all $t > t_0$. Then the substitution

$$s = \log \varphi(t), \qquad z(s) = (\varphi(t))^{-\alpha} x(t), \tag{6}$$

where $t > t_0$, converts equation (1) into the form

$$z'(s) = -(a(h(s))h'(s)+\alpha)z(s)+b_1(h(s))\lambda^\alpha h'(s)z(s-c_1)+b_2(h(s))\lambda^\alpha h'(s)z(s-c_2),$$

where $s \in J = [s_0, \infty)$. Here "$'$" means $\mathrm{d}/\mathrm{d}s$, $h(s) \equiv \varphi^{-1}(\mathrm{e}^s)$ on J, $c_1 = \log \lambda_1^{-1}$, $c_2 = \log \lambda_2^{-1}$ and $s_0 > \log \varphi(t_0)$. Then

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\left[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s)\right] = \\ &b_1(h(s))\lambda^\alpha h'(s)\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s-c_1) \\ &+ b_2(h(s))\lambda^\alpha h'(s)\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s-c_2). \end{aligned} \tag{7}$$

Due to the boundedness of $\dot{\varphi}(t)$ on I

$$\frac{1}{h'(s)} = \frac{\dot{\varphi}(h(s))}{\varphi(h(s))} = O(\mathrm{e}^{-s}) \qquad \text{as } s \to \infty.$$

From here we get

$$a(h(s))h'(s) \geq M\mathrm{e}^{(1-\beta)s} \tag{8}$$

for a suitable real $M > 0$ and every $s \geq s_0$. Then we can choose $d_0 \geq s_0$ such that $\alpha + a(h(s))h'(s) > 0$ for every $s \geq d_0$. Put $c = \min(c_1, c_2)$, $d_i = d_0 + ic$, $J_i = [d_{i-1}, d_i]$ and $M_i = \max\{|z(s)|, s \in \cup_{k=1}^i J_k\}$, $i = 1, 2, \ldots$. If we choose any $s^* \in J_{i+1}$, then we can integrate (7) over $[d_i, s^*]$ to obtain

$$\begin{aligned} &\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s)\Big|_{d_i}^{s^*} = \\ &\int_{d_i}^{s^*} b_1(h(s))\lambda^\alpha h'(s)\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s-c_1)\,\mathrm{d}s \\ &+ \int_{d_i}^{s^*} b_2(h(s))\lambda^\alpha h'(s)\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s-c_2)\,\mathrm{d}s. \end{aligned}$$

Then

$$
\begin{aligned}
z(s^*) = {} & \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} z(d_i) \\
& + \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\
& \times (\int_{d_i}^{s^*} b_1(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_1)\,\mathrm{d}s \\
& \quad + \int_{d_i}^{s^*} b_2(h(s))\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\} z(s - c_2)\,\mathrm{d}s).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(9)\ |z(s^*)| \le {} & M_i \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} \\
& + M_i \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\
& \times \int_{d_i}^{s^*} (|b_1(h(s))| + b_2(h(s))|)\lambda^\alpha h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s \\
\le {} & M_i \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} \\
& + M_i \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\
& \times \int_{d_i}^{s^*} a(h(s))h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s.
\end{aligned}
$$

Now we estimate the last integral as

$$
\begin{aligned}
& \int_{d_i}^{s^*} a(h(s))h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s \le \\
& \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\Big|_{d_i}^{s^*} + |\alpha| \int_{d_i}^{s^*} \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s.
\end{aligned}
$$

Rewrite the last term as

$$
\begin{aligned}
& |\alpha| \int_{d_i}^{s^*} \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s = \\
& \int_{d_i}^{s^*} \frac{|\alpha|}{\alpha + a(h(s))h'(s)} \frac{\mathrm{d}}{\mathrm{d}s}[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}]\,\mathrm{d}s.
\end{aligned}
$$

Notice that due to (8)

$$
\frac{|\alpha|}{\alpha + a(h(s))h'(s)} = O(\exp\{(\beta - 1)s\}) \qquad \text{as } s \to \infty.
$$

Put $\gamma = 1 - \beta > 0$. Then

$$\begin{aligned}
&\int_{d_i}^{s^*} \frac{|\alpha|}{\alpha + a(h(s))h'(s)} \frac{\mathrm{d}}{\mathrm{d}s}[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}]\,\mathrm{d}s \leq \\
&N \int_{d_i}^{s^*} \mathrm{e}^{-\gamma s} \frac{\mathrm{d}}{\mathrm{d}s}[\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}]\,\mathrm{d}s \leq \\
&N\mathrm{e}^{-\gamma d_i} \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\big|_{d_i}^{s^*}
\end{aligned}$$

for a suitable $N > 0$. Consequently,

$$\begin{aligned}
&\int_{d_i}^{s^*} a(h(s))h'(s) \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\,\mathrm{d}s \leq \\
&\exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\big|_{d_i}^{s^*}(1 + N\,\mathrm{e}^{-\gamma d_i}).
\end{aligned}$$

Substituting this back into (9) we obtain

$$\begin{aligned}
|z(s^*)| \leq\ & M_i \exp\{\alpha(d_i - s^*) - \int_{h(d_i)}^{h(s^*)} a(u)\,\mathrm{d}u\} \\
& + M_i \exp\{-\int_{s_0}^{h(s^*)} a(u)\,\mathrm{d}u - \alpha s^*\} \\
& \times \exp\{\alpha s + \int_{s_0}^{h(s)} a(u)\,\mathrm{d}u\}\big|_{d_i}^{s^*}(1 + N\,\mathrm{e}^{-\gamma d_i}) \\
\leq\ & M_i(1 + N\,\mathrm{e}^{-\gamma d_i}).
\end{aligned}$$

Consequently,

$$M_{i+1} \leq M_i(1 + N\,\mathrm{e}^{-\gamma d_i}) \leq M_1 \prod_{k=1}^{i}(1 + N\,\mathrm{e}^{-\gamma d_k}), \qquad i = 1, 2, \ldots.$$

Letting $i \to \infty$ we can see that the infinite product

$$\prod_{k=1}^{\infty}(1 + N\,\mathrm{e}^{-\gamma d_k})$$

converges. This implies that (M_i) is bounded as $i \to \infty$, hence $z(s)$ is bounded as $s \to \infty$. Substituting this back into (6) we obtain the asymptotic property (5). This completes the proof. □

Remark 2. The validity of the previous statement can be easily generalized to the case when equation (1) with m delayed arguments is considered.

Remark 3. It is easy to verify that the function $\omega(t) = \big(\varphi(t)\big)^\alpha$ occuring in (5) defines the solution of the functional equation

$$\omega\big(\tau(t)\big) = \frac{\omega(t)}{L},$$

where $\tau(t) = \max\big(\tau_1(t), \tau_2(t)\big)$, $t > t_0$.

4. Applications

In this section, we specify delays $\tau_1(t)$, $\tau_2(t)$ in (1) to illustrate our asymptotic result.

Example 1. We consider the equation

$$\dot{x}(t) = -a(t)x(t) + b_1(t)x(\lambda_1 t) + b_2(t)x(\lambda_2 t), \qquad t \in I = [0, \infty), \tag{10}$$

where $0 < \lambda_1 < \lambda_2 < 1$, $a(t)$, $b_1(t)$, $b_2(t) \in C^0(I)$. The corresponding system of Schröder's equations is

$$\begin{aligned} \varphi(\lambda_1 t) &= \lambda_1 \varphi(t), \\ \varphi(\lambda_2 t) &= \lambda_2 \varphi(t) \end{aligned}$$

and admits the identity function $\varphi(t) = t$ as the required solution. Then we can reformulate the main result as follows:

Let $a(t) \geq K/t^\beta$, $0 < |b_1(t)| + |b_2(t)| \leq La(t)$ for every $t \in I$ and suitable reals K, $L > 0$ and $\beta < 1$. If $x(t)$ is a solution of (10), then

$$x(t) = O(t^\alpha) \qquad \text{as } t \to \infty, \qquad \alpha = \frac{\log L}{\log \lambda_2^{-1}}.$$

This asymptotic estimate generalizes some parts of [7], [11] and [3]. Particularly, if we consider the equation

$$\dot{x}(t) = \beta_1(t)[x(\lambda_1 t) - x(t)] + \beta_2(t)[x(\lambda_2 t) - x(t)], \qquad t \in I \tag{11}$$

(i.e. $L = 1$), where $\beta_1(t)$, $\beta_2(t) \geq K/t^\beta$ for every $t \in I$ and suitable reals $K > 0$, $\beta < 1$, then all the solutions of (11) are bounded. We note that equation (11) with $\beta_i(t) < 0$ and constant delays has been investigated by J. Diblík [5].

Example 2. Now we investigate the asymptotic behaviour of the solutions of the equation

$$\dot{x}(t) = -a(t)x(t) + b_1(t)x(t^{\gamma_1}) + b_2(t)x(t^{\gamma_2}), \qquad t \in I = [1, \infty), \tag{12}$$

where $0 < \gamma_1 < \gamma_2 < 1$, and $a(t)$, $b_1(t)$, $b_2(t) \in C^0(I)$. It is easy to verify that the corresponding system of Schröder's equations

$$\begin{aligned} \varphi(t^{\gamma_1}) &= \gamma_1 \varphi(t), \\ \varphi(t^{\gamma_2}) &= \gamma_2 \varphi(t) \end{aligned}$$

has the solution $\varphi(t) = \log t$. Substituting this into (5) we get that if $a(t) \geq K\big/(\log t)^\beta$ and $0 < |b_1(t)| + |b_2(t)| \leq La(t)$ for every $t \in I$ and suitable reals K, $L > 0$, $\beta < 1$, then

$$x(t) = O\big((\log t)^\alpha\big) \qquad \text{as } t \to \infty, \qquad \alpha = \frac{\log L}{\log \gamma_2^{-1}}$$

for all the solutions $x(t)$ of (12).

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