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A NOTE ON DIFFERENTIAL AND INTEGRAL EQUATIONS IN LOCALLY CONVEX SPACES

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Abstract. In this survey paper we consider differential and integral equations in locally convex spaces (in particular, in these sequentially complete spaces which contain a compact barrel). We present recently obtained by us results concerning the existence and topological structure of some basic nonlinear equations and we accent applications in our results some theorems of functional analysis.

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1. Introduction

Consider the initial value problem

$$x' = f(t, x), \qquad x(0) = x_0, \tag{1}$$

where f is a bounded continuous function taking values in a quasicomplete locally convex space E. The idea to consider problem (1) in these spaces goes back to Millionščikov [13] and Hukuhara [8] who proved that (1) has a solution if the function f is compact or it satisfies the Kamke condition. The existence of solutions of (1) under different assumptions on E or f has been investigated later by many authors (see e.g. [1], [12] and [18]).

Moreover, there have appeared recently papers concerning the existence and topological structure of solutions of nonlinear integral equations in locally convex spaces (see e.g. [9] and [18]).

In Section 2 we present recently obtained by us Kneser type theorems for the equation of nth order in quasicomplete locally convex spaces. The main conditions in these results are formulated in terms of the Sadovski measure of noncompactness (see [16] for the definition and properties).

In Section 3 we consider sequentially complete locally convex spaces. Moreover, we assume that these spaces contain a compact barrel. In [1] Astala gave the following characterization of these spaces.

Lemma 1. *E is a sequentially complete locally convex space containing a compact barrel iff*

$$E = (X', \tau),$$

where X' is the dual of a barrelled normed space X and τ is a locally convex topology of X' that is stronger than the $w*$-topology but weaker than the topology of precompact convergence; briefly

$$\sigma(X', X) \leq \tau \leq \lambda(X', X).$$

By the above lemma we can use in the space E the notion of the norm.

Moreover, in [1] Astala proved that for each continuous mapping $f : [0, a] \times E \to E$, where E is as above, there exists a local solution of the problem (1). Additionally, he noted that applying the method from [17] one can prove that there exists an interval $J \subset I$ such that the set of all solutions of (1), defined on J, and considered as a subset of the space $C(J, E)$ of all continuous functions $J \to E$ with the topology of uniform convergence is compact and connected; shortly: it has the Kneser property.

Here we present results concerning the existence of continuous solutions of the nonlinear Volterra integral equation

$$x(t) = g(t) + \int_{A(t)} f(t, s, x(s))ds, \qquad t \in A, \tag{2}$$

and the Urysohn integral equation

$$x(t) = g(t) + \lambda \int_{A} f(t, s, x(s))ds, \qquad t \in A,\ \lambda \in \mathbb{R}, \tag{3}$$

considered in the space E, where $A = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$ ($a_i > 0$, $i = 1, \ldots, n$) and $A(t) = \{s \in \mathbb{R}^n : \ 0 \leq s_i \leq t_i,\ i = 1, \ldots, n\}$. In the above equations the sign "$\int$" stands for the Riemann integral.

Moreover, we characterize the topological structure of the solutions of (2) and the Darboux problem for the hyperbolic type equation.

2. Differential equation of nth order

Let E be a quasicomplete locally convex space and let $\mathcal{P}$ be a family of seminorms which generate the topology of E. Moreover, let $I = [0, a]$ be a compact interval in $\mathbb{R}$ and $B = \{x \in E : p_i(x) \leq b,\ i = 1, \ldots, k\}$, $b > 0$, $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in \mathcal{P}$.

Consider the problem

$$x^{(n)} = f(t, x) \tag{4}$$

$$x^{(j)}(0) = x_j, \quad j = 0, \ldots, n-1,$$

where $x_j \in E$ for $j = 0, \ldots, n-1$, $x_0 = 0$ and $f : I \times B \to E$ is a bounded, continuous function.

Denote by $(\beta_p(\cdot))_{p \in \mathcal{P}}$ the Sadovski measure of noncompactness. Define

$$\varphi_p(t, X) = \lim_{r \to 0^+} \beta_p(f(I_{tr} \times X)) \quad \text{for } t \in (0, a) \text{ and } X \subset B,$$

where $I_{tr} = (t - r, t + r) \cap I$ (cf. [14]). Moreover, let $B_p(0, r) = \{x \in E : \ p(x) \leq r\}$.

Theorem 1. *([3]) Assume that for every seminorm $p \in \mathcal{P}$ there exists a continuous function u_p, defined on I and such that $u_p(t) > 0$ for $t > 0$, $u_p(0) = \ldots = u_p^{(n-1)}(0) = 0$, $u_p^{(n)}(t)$ is positive , integrable in Lebesgue sense and*

$$\varphi_p(t, X) \leq \frac{u_p^{(n)}(t)}{u_p(t)} \beta_p(X) \tag{5}$$

for $t \in (0, a)$ and for every bounded set $X \subset B$, and

$$\lim_{t \to 0^+} \lim_{r \to 0^+} \frac{\beta_p(f(t, B_p(0, r)))}{u_p^{(n)}(t)} = 0. \tag{6}$$

Then there exists an interval $J = [0, d] \subset I$ such that the set of all solutions of (4), defined on J and considered as a subset of the space $C(J, E)$ is nonempty, compact and connected.

Note that the assumption (5) in Th. 2 is inspirated by the paper [7]. In the case of separable spaces Th. 2 has a simpler form, namely, the following theorem holds.

Theorem 2. *([3]) If the space E is separable, then Theorem 2 remains true, if one replaces the assumption (5) by the following one*

$$\beta_p(f(t, X)) \leq \frac{u_p^{(n)}(t)}{u_p(t)} \beta_p(X). \tag{7}$$

where $X \subset B$ is any bounded set, $t \in (0, a]$ and $p \in \mathcal{P}$.

Using another method of a proof as in the case of Th. 1 and Th. 2, namely Reichert's connectness principle from [15], the first author of this paper proved the following

Theorem 3. *([2]) In the assumptions of Th.1 instead of* (6) *assume that*

$$\lim_{t\to 0^+}\lim_{r\to 0^+} \frac{\varphi_p(t, B_p(0,r))}{u_p^{(n)}(t)} = 0.$$

Then there exists an interval $J = [0,d] \subset I$ such that the set of all solutions of (4)*, defined on J, is nonempty, compact and connected in $C(J,E)$.*

3. Nonlinear integral equations

Consider first the equation (2). Arguing similarly as in [1] we obtain the following

Theorem 4. *([4]) Assume that the functions $g : A \to E$ and $f : A^2 \times E \to E$ are continuous. Then the equation* (2) *has a local continuous solution.*

To prove the above theorem we construct the sequence of the approximate solutions of the problem (2) and applying generalized Ascoli's theorem ([10], p.81) we show that this sequence has a convergent subsequence to the solution of (2).

The following Kneser-type theorem extends Th. 4.

Theorem 5. *([5]) Under the above assumptions there exists a set*

$$J = [0,d_1] \times [0,d_2] \times \ldots \times [0,d_n] \subset A$$

such that the set S of all continuous solutions of (2)*, defined on J, is nonempty, compact and connected in the space $C(J,E)$.*

To prove Th. 5 one can not apply the method from [17]. Now, we sketch the idea of the proof of Th. 5. Let r be any positive number. Since the ball $B_r = \{x \in E : \ \|x\| \le r\}$ is convex, ballanced, closed, bounded and sequentially complete, in view of the Banach-Mackey theorem ([10], p.91) it is absorbing by the barrel and therefore it is compact. Hence for every number $r > 0$ there exists a number $m_r > 0$ such that

$$\|f(t,s,x)\| \le m_r \quad \text{for} \quad (t,s) \in A \quad \text{and} \quad x \in B_r$$

(cf. Lemma 1). Now, knowing that f is locally bounded we can define $J = [0,d_1] \times [0,d_2] \times \ldots \times [0,d_n]$ in the classical way. Denote by $\widetilde{B}$ the set of all continuous functions $J \to B_b$, where b is the suitably choosen number. We consider $\widetilde{B}$ as a subspace of $C(J,E)$. Set

$$G(x)(t) = g(t) + \int\limits_{A(t)} f(t,s,x(s))ds, \qquad t \in J,\ x \in \widetilde{B}.$$

One can easy show that $G(\widetilde{B}) \subset \widetilde{B}$ and the family $G(\widetilde{B})$ is equiuniformly continuous. Moreover, in view of the Krasnoselski-Krein-type lemma (cf. [11]) we deduce that G is continuous.

For any $\varepsilon > 0$ denote by S_ε the set of all $x \in \widetilde{B}$ such that $\|x(t) - G(x)(t)\| < \varepsilon$ for every $t \in J$. It can be proved (cf. [6]) that for sufficiently small $\varepsilon > 0$, the set S_ε is nonempty and connected. Using this fact, the generalized Ascoli theorem and the continuity of G we infer that S is nonempty and compact.

To prove that S is connected it is enough to apply standard arguments as e.g. in [6].

Now, we pass to the equation (3). As in the above theorems we assume that the functions $g : A \to E$ and $f : A^2 \times E \to E$ are continuous. Our next result is the following

Theorem 6. *([5]) Under the above assumptions there exists $\eta > 0$ such that for $\lambda \in \mathbb{R}$ with $|\lambda| < \eta$, the equation (3) has a continuous solution defined on A.*

Analogously as in the proof of Th. 5 we deduce that f is locally bounded, next we define η and the subset $\widetilde{B} \subset C(A, E)$ in the classical way. Put

$$G(x)(t) = g(t) + \lambda \int\limits_A f(t, s, x(s))ds, \quad t \in A, \quad x \in \widetilde{B}.$$

The operator G maps continuously $\widetilde{B}$ into itself. Let $V = \overline{\text{conv}} G(\widetilde{B})$. By the generalized Ascoli theorem we deduce that V is compact and we can apply the Schauder-Tychonoff theorem for the mapping $G|_V$.

Now, let pass on to the Darboux problem for the hyperbolic partial differential equation.

Let $B = \{z \in E : \|z\| \le b\}$, $A = [0, a_1] \times [0, a_2]$ $(a_1, a_2 > 0)$ and let $f : A \times B \to E$ be a continuous mapping. Again, by the Banach-Mackey theorem the mapping f is norm-bounded on $A \times B$. In view of this, we choose a subrectangle $J = [0, d_1] \times [0, d_2]$ in the classical way and consider the following Darboux problem

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z), \qquad (x, y) \in J,$$

$$z(x, 0) = 0, \quad 0 \le x \le d_1, \qquad z(0, y) = 0, \quad 0 \le y \le d_2. \tag{8}$$

It can be easily seen that the problem (8) is equivalent to the following integral equation

$$z(x, y) = \int\limits_0^x \int\limits_0^y f(\xi, \eta, z(\xi, \eta))d\xi d\eta, \quad (x, y) \in J,$$

where the sign"$\int\int$" stands for the Riemann integral. In view of this equivalence, as a corollary from Th. 5 we obtain the following Kneser-type characterization for the problem (8).

Theorem 7. *([5]) Under the above assumptions the set of all solutions of* (8), *defined on* J, *is nonempty, compact and connected in* $C(J, E)$.

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