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ON EXISTENCE OF SINGULAR SOLUTIONS OF N-TH ORDER DIFFERENTIAL EQUATIONS

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Abstract. In the paper sufficient conditions are given under which the equation $y^{(n)} = f(t, y, \dots, y^{(n-2)})g(y^{(n-1)})$ has a singular solution $y : [T, \tau) \to \mathbb{R}, \tau < \infty$ fulfilling $\lim_{t\to\tau_-} y^{(i)}(t) = c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-2$ and $\lim_{t\to\tau_-} |y^{(n-1)}(t)| = \infty$.

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Consider the n-th order differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-2)})g(y^{(n-1)}) \tag{1}$$

where $n \geq 2, f \in C^o(\mathbb{R}_+ \times \mathbb{R}^{n-1}), g \in C^o(\mathbb{R}), \mathbb{R}_+ = [0, \infty), \mathbb{R} = (-\infty, \infty)$, there exists $\alpha \in \{-1, 1\}$ such that

$$\alpha f(t, x_1, \dots, x_{n-1})x_1 > 0 \quad \text{for} \quad x_1 \neq 0 \quad \text{and} \quad g(x) \geq 0 \quad \text{for} \quad x \in \mathbb{R}. \tag{2}$$

Hence, (1) fulfills the sign condition.

A solution y defined on $[T, \tau) \subset \mathbb{R}_+$ is called singular if $\tau < \infty$ and y cannot be defined for $t = \tau$. A singular solution y is called nonoscillatory if $y \neq 0$ in a left neighbourhood of τ, otherwise it is called oscillatory.

The problem of the existence of a nonoscillatory singular solution y of (1) fulfilling

$$y^{(i)}(t)y(t) > 0, i = 0, 1, \dots, n-1 \tag{3}$$

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in a left neighbourhood of τ is posed and studied in [5,6] (in case $\alpha = 1$) for Emden-Fowler equation

$$y^{(n)} = r(t)|y|^{\lambda} \operatorname{sgn} y, \quad r \neq 0, \tag{4}$$

see [1] and [2], too. For Eq. (1) the results are generalized in [7,8]. The existence of oscillatory singular solution is proved only for Eq. (4) in [3]. Note that singular solutions of (4) (with all derivatives) are unbounded, see e.q. [9].

On the other hand singular solutions with different asymptotic behaviour than (3) may exist. Jaroš and Kusano announced that in [4] they studied a special case of (1), the second order equation

$$y'' = r(t)|y|^{\sigma}|y'|^{\lambda} \operatorname{sgn} y, \quad \sigma > 0, r < 0 \quad \text{on} \quad \mathbb{R}_+.$$

They proved that the necessary and sufficient condition for the existence of a singular solution y fulfilling

$$\lim_{t \to \tau_-} y(t) = c \in [0, \infty), \; \lim_{t \to \tau_-} y'(t) = -\infty \tag{5}$$

is $\lambda > 2$; solutions fulfilling (5) are called black hole solutions.

In our paper we generalize this result for (1).

We will study the existence of a singular solution y fulfilling the conditions:

$$\begin{gathered} \tau \in (0, \infty), \lim_{t \to \tau_-} y^{(i)}(t) = c_i \in \mathbb{R}, \; i = 0, 1, \dots, n-2, \\ \lim_{t \to \tau_-} |y^{(n-1)}(t)| = \infty. \end{gathered} \tag{6}$$

This solution is nonoscillatory. Moreover the sign of $y^{(n-1)}, \alpha$ and c_0 cannot be arbitrary.

Lemma 1. *Let y be a solution of* (1) *fulfilling* (6).
(a) If $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$ *then* $\alpha c_0 \geq 0$.
(b) If $\lim_{t \to \tau_-} y^{(n-1)}(t) = -\infty$ *then* $\alpha c_0 \leq 0$.

Proof. (a) Let $\alpha = 1$ for simplicity and suppose $c_0 < 0$. Then according to (1) and (2) $y^{(n)}(t) \leq 0$ for large t that contradicts $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$. Hence $c_0 \geq 0$.
(b) The proof is similar.

Denote by $[[x]]$ the entire part of x.

Theorem 1. *Let $\tau \in (0, \infty), \lambda > 2, c_0 \neq 0, c_i \in \mathbb{R}$ for $i = 1, \dots, n-2$ and $M \in (0, \infty)$. Let $\beta = \alpha \operatorname{sgn} c_0$ and*

$$g(x) \geq |x|^{\lambda} \qquad for \qquad \beta x \geq M. \tag{7}$$

Then there exists a singular solution y of (1) *fulfilling* (6) *that is defined in a left neighbourhood of τ.*

If, moreover, $\varepsilon > 0$,

$$n + \frac{1-\alpha}{2} \quad \textit{is odd}, \quad (-1)^i c_i c_0 \ge 0 \quad \textit{for} \quad i = 1, 2, \dots, n-2 \tag{8}$$

and

$$\left| \int_0^{\beta\varepsilon} \frac{ds}{g(s)} \right| = \infty \tag{9}$$

then y *is defined on* $[0, \tau)$.

Proof. We prove the statement for $\alpha = 1$ and $c_0 > 0$ (thus $\beta = 1$). For the other cases the proof is similar.

Let $N > 2\max(c_0, |c_1|, \dots, |c_{n-2}|)$. Consider the auxilliary problem

$$\begin{gathered} y^{(n)} = f\left(t, \chi_0, (y), \chi(y'), \dots, \chi(y^{(n-2)})\right) g(y^{(n-1)}), \\ y^{(i)}(\tau) = c_i, i = 0, 1, \dots, n-2, \quad y^{(n-1)}(\tau) = k \end{gathered} \tag{10}$$

where $k \in \{k_0, k_0 + 1, \dots\},\ k_0 \ge [[2M]]$,

$$\begin{aligned} \chi_0(s) &= s && \text{for} \quad \tfrac{c_0}{2} \le s \le N, \\ &= N && \text{for} \quad s > N, \\ &= c_0/2 && \text{for} \quad s < c_0/2, \\ \chi(s) &= s && \text{for} \quad |s| \le N, \\ &= N && \text{for} \quad s > N, \\ &= -N && \text{for} \quad s < -N. \end{aligned} \tag{11}$$

Denote by y_k a solution of (10) and by J_1 the penetration of its definition interval and $[0, \tau]$. Note, that (2), (10) and (11) yield

$$y_k^{(n)}(t) \ge 0 \quad \text{on} \quad J_1. \tag{12}$$

Put

$$\begin{aligned} M_1 = &\min\{f(t, x_1, \dots, x_{n-1}) : t \in [0, \tau], \frac{c_0}{2} \le x_1 \le N, \\ &|x_j| \le N, j = 2, \dots, n-1\} > 0, \\ M_2 = &\max\{f(t, x_1, \dots, x_{n-1}) : t \in [0, \tau], \frac{c_0}{2} \le x_1 \le N, \\ &|x_j| \le N, j = 2, \dots, n-1\}, \\ M_3 = &[(\lambda - 1)M_1]^{-\frac{1}{\lambda - 1}}. \end{aligned}$$

Further, let $J = [T, \tau] \subset J_1$ be such that $T < \tau$,

$$\sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!} (\tau - T)^{j-i} + \frac{\lambda - 1}{\lambda - 2} M_3 (\tau - T)^{n-i-1-\frac{1}{\lambda-1}} \le N, \quad i = 0, 1, \dots, n-2, \tag{13}$$

$$\sum_{j=1}^{n-2} \frac{|c_j|}{j!} (\tau - T)^j + \frac{\lambda - 1}{\lambda - 2} M_3 (\tau - T)^{n-1-\frac{1}{\lambda-1}} \le \frac{c_0}{2} \tag{14}$$

and

$$M_2(\tau - T) < \int_M^{2M} \frac{ds}{g(s)}. \tag{15}$$

As (7), $\lambda > 2$ and $n \geq 2, J$ exists.

We prove that

$$y_k^{(n-1)}(t) \geq M, t \in J. \tag{16}$$

Suppose, contrarily, that $T_1 \in [T, \tau)$ exists such that $y_k^{(n-1)}(T_1) = M$. Then with respect to (10) and (12) $y_k^{(n-1)}(t) \geq M$ for $t \in [T_1, \tau]$. From this and from (10) and (11)

$$y_k^{(n)}(t) \leq M_2 g\left(y_k^{(n-1)}(t)\right), t \in [T_1, \tau]$$

and hence, by the integration on $[T_1, \tau]$,

$$\int_M^{2M} \frac{ds}{g(s)} \leq \int_M^k \frac{ds}{g(s)} \leq M_2(\tau - T_1) \leq M_2(\tau - T).$$

The contradiction with (15) proves that $y^{(n-1)} \neq M$ for $t \in J$. From this, from (12) and $y_k^{(n-1)}(\tau) = k > M$ (16) holds.

Further, (7), (10), (11) and (16) yield

$$y_k^{(n)}(t) \geq M_1 g\left(y^{(n-1)}(t)\right) \geq M_1 \left(y^{(n-1)}(t)\right)^\lambda, \ t \in J$$

and by the integration on $[t, \tau] \subset J$ we have

$$\begin{aligned} &(y_k^{(n-1)}(t))^{1-\lambda} - k^{1-\lambda} \geq M_1(\lambda - 1)(\tau - t), \\ &y_k^{(n-1)}(t) \leq M_3(\tau - t)^{-\frac{1}{\lambda-1}}, \ t \in [T, \tau), \ k \geq k_0. \end{aligned} \tag{17}$$

Hence, using the Taylor series formula at τ, (13), (17) and $\lambda > 2$, we have

$$\begin{aligned} \left|y_k^{(i)}(t)\right| &\leq \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - t)^{j-i} + \left|\int_\tau^t \frac{(t-s)^{n-i-2}}{(n-i-2)!} \ y_k^{(n-1)}(s) ds\right| \leq \\ &\leq \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - T)^{j-i} + \frac{M_3(\tau - t)^{n-i-2}}{(n-i-2)!} \left|\int_\tau^t (\tau - s)^{-\frac{1}{\lambda-1}} ds\right| \\ &\leq N \qquad , i = 0, 1, \ldots, n-2, t \in [T, \tau), k \geq k_0. \end{aligned}$$

Similarly, using (14) and (17)

$$y_k(t) \geq c_0 - \sum_{j=1}^{n-2} \frac{|c_j|}{j!}(\tau - T)^j - \frac{\lambda - 1}{\lambda - 2} M_3 (\tau - T)^{n-1-\frac{1}{\lambda-1}} \geq \frac{c_0}{2},$$

$$t \in [T, \tau), \ k \geq k_0.$$

From these estimations and (11) we can see that y_k is the solution of Eq. (1), too. Moreover, the sequences $\{y_k^{(i)}\}_{k_0}^{\infty}, i = 0, \dots, n-1$ are uniformly bounded and equipotentially continuous on every segment of $[T, \tau)$. Hence according to Arzel-Ascoli Theorem there exists a subsequence that converges uniformly to a solution y of (1). Evidently, the conditions (6) are fulfilled with $\lim_{t\to\tau_-} y^{(n-1)}(t) = \infty$.

Let (8) and (9) be valid. Let the above given solution y be defined on $(\bar\tau, \tau) \subset [0, \tau)$ and cannot be extended to $t = \bar\tau$. Then

$$\limsup_{t\to\bar\tau_+} \left| y^{(n-1)}(t) \right| = \infty. \tag{18}$$

First, we prove that

$$y^{(n-1)}(t) > 0 \quad \text{on} \quad (\bar\tau, \tau). \tag{19}$$

Thus, suppose that there exists $\tau_1 \in (\bar\tau, \tau)$ such that $y^{(n-1)}(\tau_1) = 0$ and $y^{(n-1)}(t) > 0$ on (τ_1, τ). It follows from this and from (6) that $y^{(j)}, j = 0, 1, \dots, n-2$ are bounded, $|y^{(j)}(t)| \le K, j = 0, 1, \dots, n-2, t \in [\tau_1, \tau)$. Let $\tau_2 \in (\tau_1, \tau)$ be such that $y^{(n-1)}(\tau_2) = \varepsilon$. Then by the integration of (1) and by (9)

$$\infty = \int_0^{\varepsilon} \frac{ds}{g(s)} = \int_{\tau_1}^{\tau_2} f\left(t, y(t), \dots, y^{(n-2)}(t)\right) dt < \infty.$$

Hence, (19) is valid, and (8) and (19) yield $y(t) > 0$ on $(\bar\tau, \tau)$. From this and from (1) $y^{(n)}(t) > 0$ on $(\bar\tau, \tau)$, that, together with (19), contradicts (18). Thus y is defined at $t = \bar\tau$ and $\bar\tau = 0$.

Corollary 1. *Let $\lambda > 2$ and $M \in \mathbb{R}_+$ be such that*

$$g(x) \ge x^{\lambda} \quad \text{for} \quad x \ge M.$$

Then (1) *has a singular solution.*

Remark 1. For $\alpha = 1$ the conclusion of Corollary 1 is known, see, e.g., [9, Theorem 11.3].

The following result shows that for the existence of a singular solution with (6) λ cannot be equal to 2.

Theorem 2. *Let $M \in (0, \infty)$ be such that $g(x) \le x^2$ for $|x| \ge M$. Then Eq.* (1) *has no singular solution y fulfilling* (6).

Proof. Let y be singular and fulfil (6). Suppose, for simplicity, $\alpha = 1$ and $\lim_{t\to\tau_-} y^{(n-1)}(t) = \infty$. From this there exists a left neighbourhood $[\tau_1, \tau)$ of τ such that $|y^{(i)}(t)| \le M_1 < \infty$ for $i = 0, 1, \dots, n-2$ and $y^{(n-1)}(t) \ge M$ on $[\tau_1, \tau)$

where M_1 is a suitable constant. Hence, using the assumptions of the theorem we have

$$\infty = ln\ \frac{y^{(n-1)}(\tau)}{y^{(n-1)}(\tau_1)} = \int\limits_{\tau_1}^{\tau} \frac{y^{(n)}(s)}{y^{(n-1)}(s)} ds \le \int\limits_{\tau_1}^{\tau} \left| f\left(s, y(s), \dots, y^{(n-2)}(s)\right)\right| y^{(n-1)}(s) ds$$
$$\le \left(c_{n-2} - y^{(n-2)}(\tau_1)\right) \max |f(s, x_1, \dots, x_{n-1})| < \infty$$

where the maximum is taken for $s \in [\tau_1, \tau], |x_i| \le M_1, i = 1, \dots, n-1$. The contradiction proves the conclusion.

Corollary 2. *Let $c_0 \neq 0, M \in (0, \infty)$ and $g(x) = |x|^\lambda$ for $|x| \ge M$. Then* (1) *has a singular solution y fulfilling* (6) *if and only if $\lambda > 2$.*

Proof. It follows from Theorems 1 and 2.

Remark 2. Note, that, especially, eq.

$$y^{(n)} = f(t, y, y', \dots, y^{(n-2)})$$

has no singular solutions satisfying (6).

In the next part of the paper the case $c_0 = 0$ will be investigated.

Theorem 3. *Let $\beta \in \{-1, 1\}, \sigma > 0$, $\varepsilon > 0, \tau \in (0, \infty)$, $M \in (0, \infty), \alpha \in \{-1, 1\}$*

$$\lambda > \sigma(n-2) + 2, \tag{20}$$

$$c_0 = 0, (-1)^i \beta\ c_i \ge 0 \quad for \quad i = 1, 2, \dots, n-2, \tag{21}$$

and

$$n + \frac{1-\alpha}{2} \quad be\ odd. \tag{22}$$

Let (7) *hold and a continuous function $r : \mathbb{R}_+ \to \mathbb{R}$ exist such that*

$$\alpha r(t) > 0 \quad on \quad R_+,$$
$$|f(t, x_1, x_2, \dots, x_{n-1})| \ge |r(t)|\, |x_1|^\sigma \quad for \quad t \in [0, \tau],$$
$$\beta x_1 \in [0, \varepsilon], (-1)^j \beta x_{j+1} \in \left[(-1)^j \beta c_j, (-1)^j \beta c_j + \varepsilon\right], \ j = 1, 2, \dots, n-2.$$

Then there exists a singular solution y of (1) *fulfilling* (6) *that is defined in a left neighbourhood of τ. If, moreover,* (9) *holds, then y is defined on $[0, \tau)$.*

Proof. Let $\alpha = 1$ and $\beta = 1$; thus n is odd. For the other cases the proof is similar. Put for $i \in \{0, 1, \dots, n-2\}$

$$\begin{aligned} \chi_i(s) &= s && \text{for} \quad (-1)^i c_i \le (-1)^i s \le (-1)^i c_i + \varepsilon, \\ &= c_i + (-1)^i \varepsilon && \text{for} \quad (-1)^i s > (-1)^i c_i + \varepsilon, \\ &= c_i && \text{for} \quad (-1)^i s < (-1)^i c_i. \end{aligned} \tag{23}$$

Consider the Cauchy problem

$$\begin{aligned} y^{(n)} &= f\left(t, \chi_0(y), \chi_1(y'), \dots, \chi_{n-2}(y^{(n-2)})\right) g\left(y^{(n-1)}\right), \\ y^{(i)}(\tau) &= c_i, i = 0, 1, \dots, n-2, \quad y^{(n-1)}(\tau) = k \end{aligned} \tag{24}$$

where $k \in \{k_0, k_0 + 1, \dots\}, k_0 \geq [[2M]]$.

Denote by y_k a solution of (24) and J_1 the penetration of its definition interval and $[0, \tau]$. Note, that $\alpha = 1$, (23), (24) yield

$$y_k^{(n)}(t) \geq 0 \quad \text{and} \quad y_k^{(n-1)} \quad \text{is nondecreasing on} \quad J_1. \tag{25}$$

Put $M_1 = \frac{1}{[(n-1)!]^\sigma} \min_{t \in [0,\tau]} r(t) > 0,\ M_2 = \left[\frac{M_1}{\sigma(n-1)+1}(\lambda + \sigma - 1)\right]^{-\frac{1}{\lambda+\sigma-1}}$,

$$\sigma_1 = \frac{\sigma(n-1)+1}{\lambda + \sigma - 1}, M_3 = \max f(t, x_1, \dots, x_{n-1}),$$

where the maximum is given for $t \in [0, \tau], 0 \leq x_1 \leq \varepsilon, (-1)^i c_i \leq (-1)^i x_{i+1} \leq (-1)^i c_i + \varepsilon, i = 1, \dots, n-2$. Then (20) yields $\sigma_1 \in (0, 1)$.

Further, let $J = [T, \tau] \subset J_1$ be such that $T < \tau$,

$$\begin{aligned} \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - T)^{j-i} + \frac{M_2}{(n-i-2)!(1-\sigma_1)}(\tau - T)^{n-i-\sigma_1-1} &\leq (-1)^i c_i + \varepsilon \\ i = 0, 1, \dots, n-2 \end{aligned} \tag{26}$$

and

$$M_3(\tau - T) < \int_M^{2M} \frac{ds}{g(s)}. \tag{27}$$

Using (27), it can be proved similarly to the proof of Theorem 1, that (16) holds. Hence, using (21) and (22) we have

$$(-1)^i y_k^{(i)}(t) \geq (-1)^i c_i \geq 0 \quad \text{on} \quad J,\ i = 0, 1, 2, \dots, n-2. \tag{28}$$

The Taylor series formula at $t = \tau$, (16), (21), (25) and n be odd yield

$$\begin{aligned} y_k(t) &= \sum_{j=0}^{n-2} c_j \frac{(t-\tau)^j}{j!} + \int_\tau^t \frac{(t-s)^{n-2}}{(n-2)!} y_k^{(n-1)}(s) ds \geq \int_\tau^t \frac{(t-s)^{n-2}}{(n-2)!} y_k^{(n-1)}(s) ds \\ &\geq \frac{(\tau - t)^{n-1}}{(n-1)!} y_k^{(n-1)}(t), \qquad t \in J, \end{aligned}$$

and from (24), (16), (25), (28) and the assumptions of the theorem

$$y_k^{(n)}(t) \geq r(t)(y_k(t))^\sigma [y_k^{(n-1)}(t)]^\lambda \geq M_1(\tau - t)^{\sigma(n-1)} \left(y_k^{(n-1)}(t)\right)^{\lambda+\sigma}, t \in J.$$

Hence, by the integration on $[t, \tau]$ we obtain similarly to the proof of Theorem 1

$$y_k^{(n-1)}(t) \le M_2(\tau - t)^{-\sigma_1}, t \in [T, \tau), k = k_0, k_0 + 1, \dots$$

From this, using the Taylor series formula at $t = \tau$, (26), (28) and $\sigma_1 < 1$ we have

$$0 \le (-1)^i c_i \le (-1)^i y_k^{(i)}(t) = \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - t)^{j-i} +$$

$$(-1)^i \int_\tau^t \frac{(t-s)^{n-i-2}}{(n-i-2)!}\, y_k^{(n-1)}(s) ds$$

$$\le \sum_{j=i}^{n-2} \frac{|c_j|}{(j-i)!}(\tau - t)^{j-i} + \frac{M_2(\tau - t)^{n-i-1-\sigma_1}}{(n-i-2)!(1-\sigma_1)} \le (-1)^i c_i + \varepsilon,$$

$$i = 0, 1, \dots, n-2.$$

Thus, according to (23), y_k is a solution of Eq. (1), too and the rest of the proof is similar as in Theorem 1.

The following theorem shows that the condition (20) cannot be weaken.

Theorem 4. *Let* $c_i = 0, i = 0, 1, \dots, n-2, \sigma > 0, n \ge 2, n + \frac{1-\alpha}{2}$ *be odd,* $\alpha \in \{-1, 1\}$ *and let* $r \in C^0(\mathbb{R}_+), \alpha r > 0$ *on* $\mathbb{R}_+$. *Then the equation*

$$y^{(n)} = r(t)|y|^\sigma |y^{(n-1)}|^\lambda \operatorname{sgn} y \tag{29}$$

has a singular solution y *fulfilling* (6) *if, and only if* $\lambda > \sigma(n-2) + 2$.

Proof. In view of Theorem 3 we must prove the necessity only. Let $\lambda \le \sigma(n-2)+2$, y be singular and fulfilling (6). Suppose, for simplicity, that $r > 0$, $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$ and thus n be odd. In the other cases the proof is similar. Then there exists $t_0 \in [0, \tau)$ such that

$$(-1)^i y^{(i)}(t) > 0,\ i = 0, 1, \dots, n-2,\ y^{(n-1)}(t) \ge 1,\ y^{(n)}(t) \ge 0 \quad \text{on} \quad J = [t_0, \tau). \tag{30}$$

Then using the Taylor series formula on $[t, \tau]$ and (6) we obtain

$$y(t) = \int_\tau^t \frac{(t-s)^{n-2}}{(n-2)!} y^{(n-1)}(s) ds \le \frac{(\tau - t)^{n-2}}{(n-2)!} |y^{n-2}(t)|, t \in J. \tag{31}$$

Further,

$$|y^{(n-2)}(t)| = \int_t^\tau y^{(n-1)}(s) ds \ge y^{(n-1)}(t)(\tau - t), t \in J$$

and hence, using (31)

$$y(t)[y^{(n-1)}(t)]^{n-2} \le \frac{[y^{(n-2)}(t)]^{n-1}}{(n-2)!} \le M_1, t \in J$$

where M_1 is a suitable number. From this,(30) and from $\lambda \le \sigma(n-2)+2$

$$\infty = \ln \frac{y^{(n-1)}(\tau)}{y^{(n-1)}(t_0)} = \int_{t_0}^{\tau} \frac{y^{(n)}(s)}{y^{(n-1)}(s)} ds = \int_{t_0}^{\tau} r(s)\ y^{\sigma}(s)[y^{(n-1)}(s)]^{\lambda-1} ds \le$$
$$\le M_1^{\sigma} \int_{t_0}^{\tau} r(s)[y^{(n-1)}(s)]^{\lambda-1-\sigma(n-2)} ds$$
$$\le M_1^{\sigma} \int_{t_0}^{\tau} r(s) y^{(n-1)}(s) ds \le M_1^{\sigma} \max_{0\le s\le \tau} r(s) \left| y^{(n-2)}(t_0) \right| < \infty.$$

The contradiction proves the conclusion.

The following proposition shows that condition (22) in Theorem 3 cannot be weaken.

Proposition 1. *Let $\beta \in \{-1, 1\}$, (21), $c_{n-2} = 0$ and $n + \frac{1-\alpha}{2}$ be even. Then* (1) *has no singular solution fulfilling* (6).

Proof. Let for the simplicity $\alpha = 1$ and $\beta = -1$; for the other cases the proof is similar. Hence, n is even. Let y be a singular solution of (1) fulfilling (6). Then (1) and (21) yield $y(t) < 0, y^{(n-1)}(t) > 0$. Thus $y^{(n)}(t) > 0$ in a left neighbourhood J of τ that contradicts (1), (2) and $\alpha = 1$.

Remark 3. The following conclusion follows from Corollary 2 and Theorem 4. *Let $n = 2$. Then Eq.* (29) *has a singular solution y, fulfilling* (6) *if, and only if $\lambda > 2$.* Hence our results generalize the above mentioned one of Jaroš and Kusano.

Open problem. It is possible to look for sufficient and (or) necessary conditions under which there is a singular solution y of (1) satisfying

$$\tau \in (0,\infty), \quad k \in \{0, 1, \dots, n-2\},$$
$$\lim_{t\to\tau_-} y^{(i)}(t) = c_i \in \mathbb{R}, \quad i = 0, 1, \dots, k,$$
$$\lim_{t\to\tau_-} |y^{(j)}(t)| = \infty, \quad j = k+1, \dots, n-1.$$

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