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DERIVATION OF THE REYNOLDS EQUATION FOR LUBRICATION OF A ROTATING SHAFT

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Abstract. In this paper, using the asymptotic expansion, we prove that the Reynolds lubrication equation is an approximation of the full Navier–Stokes equations in thin gap between two coaxial cylinders in relative motion. Boundary layer correctors are computed. The error estimate in terms of domain thickness for the asymptotic expansion is given. The corrector for classical Reynolds approximation is computed.

1. Introduction

We study the lubrication process of a slipper bearings. A circular shaft of radius R and length l rotates on lubricated support with angular velocity ω. Between the shaft and the support there is a thin domain, of thickness $\varepsilon \ll l$, completely filled with a viscous incompressible fluid (lubricant) injected by some prescribed velocity. Our goal is to find the effective equations governing the flow of that thin liquid film.

We start from the Navier-Stokes system describing the microscopic flow of a viscous fluid in thin domain between two coaxial cylinders in relative motion. Unlike in [7], where the technique of two scale asymptotic expansion has been used only in a formal way, we derive rigorously the basic equations for hydrodynamic lubrication with a viscous incompressible fluid. Performing a precise asymptotic analysis of this singularly perturbed problem we study the behaviour of the flow as $\varepsilon \to 0$. At the limit, we find the classical Reynolds equations governing the 2-dimensional macroscopic flow, as an approximation of the Navier-Stokes system in thin 3-dimensional domain. Using the boundary layer correctors we prove, not only the convergence of the Navier-Stokes velocity and pressure towards their 2-dimensional approximations, but we also find the order of accuracy for Reynolds model.

The study of lubrication problems goes back to the celebrated work of Reynolds [13] published in 1886. He studied the thin film flow in a rather heuristic manner

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and did not give any relation between his model and the Navier–Stokes equations. The formal relation between Navier–Stokes equations in a thin domain and the Reynolds equations, using asymptotic expansions, was given in Elrod [7], Capriz [5] and Wannier [14].

The rigorous mathematical justification of the Reynolds equation for a flow between two plain (and not curved as in our case) surfaces was given in Bayada and Chambat [2] and Cimatti [6]. However, those authors prove only the weak convergence of the linearized, Stokes, flow to the Reynolds flow. Such weak convergence on rescaled domain, frequently used for justification of lower-dimensional approximations, justifies the Reynolds model but in a rather weak way. It does not give the order of accuracy for Reynolds approximation. A precise study of asymptotic behaviour of the viscous flow in a thin domain was given by Nazarov[12], but in an infinite [1] thin layer between with two fixed, plain surfaces[2]. The contribution of the present paper is that we study the problem with a curved geometry (corresponding to the real-life situation) and we estimate the difference between the solution of the Navier-Stokes system in thin domain and the solution of the Reynolds system in terms of the domain thickness. Furthermore, we give the detailed study of the boundary layer appearing at the ends of cylinders where the lubricant is being injected. Finally we give the corrector for the Reynolds model giving the higher order of accuracy.

To finish the introduction we mention some references related to our problem. An interesting study of weak inertial effects can be found in [1]. A nonlinear model describing the strong inertial effects for a fast flow through a rough thin domain was justified in Bourgeat and Marušić-Paloka [3], [4].

Flow through a thin curved domain was treated in Marušić-Paloka [11], with a special reference to the effects of flexion and torsion of the domain, but only in the case of tubular domain, leading to the 1-dimensional model.

2. The Problem

To describe the geometry of the film we use the cylindrical coordinates (r, φ, z). We denote by $\Xi : \mathbf{R}^3 \to \mathbf{R}^3$ the change of variable $\Xi(x_1, x_2, x_3) = (r, \varphi, z)$ where (x_1, x_2, x_3) are the cartesian and (r, φ, z) are the cylindrical coordinates of a point x. We suppose that the film thickness is εh, where $h = h(\varphi)$ and ε is a small parameter. The film is an open set

$$\mathcal{C}_\varepsilon = \{\Xi^{-1}(r, \varphi, z) \in \mathbf{R}^3 \; ; \; \varphi \in]0, 2\pi[, z \in]0, l[\, , R < r < R + \varepsilon h(\varphi)\} \, ,$$

where the function $h :]0, 2\pi[\to \mathbf{R}^+$ is of class C^2, 2π-periodic and $0 < \beta_1 \le h(\varphi) \le \beta_2$, $\varphi \in]0, 2\pi[$. The flow in domain $\mathcal{C}_\varepsilon$ is governed by the Navier–Stokes system

[1] to avoid the boundary layer effects on the edge of domain

[2] that are not in relative motion, as in our case

$$
(1) \qquad \begin{cases} -\mu\Delta u^\varepsilon + (u^\varepsilon\nabla)u^\varepsilon + \nabla p^\varepsilon = 0 & \text{in } \mathcal{C}_\varepsilon\,, \\ \text{div } u^\varepsilon = 0 & \text{in } \mathcal{C}_\varepsilon\,, \\ u^\varepsilon = 0 & \text{for } r = R + \varepsilon h(\varphi)\,, \\ u^\varepsilon = \omega \vec{e}_\varphi & \text{for } r = R\,, \\ u^\varepsilon = g_0(\frac{r-R}{\varepsilon}, \varphi) & \text{for } z = 0\,, \\ u^\varepsilon = g_l(\frac{r-R}{\varepsilon}, \varphi) & \text{for } z = l\,, \end{cases}
$$

where p^ε and u^ε are the pressure and the velocity. In order to have a well posed problem we suppose that the functions $g_\alpha \in C^2(\mathcal{S}_1)$, $\alpha = 0, l$, $\mathcal{S}_1 = \{(\rho, \varphi); \rho \in]0, h(\varphi)[, \varphi \in]0, 2\pi[\}$ are 2π-periodic in φ and satisfy the hypothesis

(H1): $g_\alpha(h(\varphi), \varphi) = 0, \;\; g_\alpha(0, \varphi) = \omega\vec{e}_\varphi\ , \ \alpha = 0, l,$

(H2): $\int_0^{2\pi}\int_0^{h(\varphi)} \rho\vec{e}_z \cdot g_0(\rho,\varphi)d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi)} \rho\vec{e}_z \cdot g_l(\rho,\varphi)d\rho d\varphi\ ,$

(H3): $\int_0^{2\pi}\int_0^{h(\varphi)} \vec{e}_z \cdot g_0(\rho,\varphi)d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi)} \vec{e}_z \cdot g_l(\rho,\varphi)d\rho d\varphi.$

The classical result shows that for each $\varepsilon > 0$:

Theorem 1. *Under the assumptions* (**H1**),(**H2**) *and* (**H3**) *the problem* (1) *has a solution* $(u^\varepsilon, p^\varepsilon) \in H^1(\mathcal{C}_\varepsilon)^3 \times L^2(\mathcal{C}_\varepsilon)/\mathbf{R}$.

For the proof see e.g. Galdi [8].

3. Asymptotic expansion

Due to the geometry of the domain it is natural to work in the cylindrical coordinate system. In the cylindrical coordinates the Navier–Stokes equations read (see e.g. [10]):

$$
\begin{aligned}
&-\mu(\Delta u_r^\varepsilon - \frac{u_r^\varepsilon}{r^2} - \frac{2}{r^2}\frac{\partial u_\varphi^\varepsilon}{\partial\varphi}) + u_r^\varepsilon\frac{\partial u_r^\varepsilon}{\partial r} + \frac{u_\varphi^\varepsilon}{r}\frac{\partial u_r^\varepsilon}{\partial\varphi} + u_z^\varepsilon\frac{\partial u_r^\varepsilon}{\partial z} - \frac{(u_\varphi^\varepsilon)^2}{r} + \frac{\partial p^\varepsilon}{\partial r} = 0\,, \\
&-\mu(\Delta u_\varphi^\varepsilon - \frac{u_\varphi^\varepsilon}{r^2} + \frac{2}{r^2}\frac{\partial u_r^\varepsilon}{\partial\varphi}) + u_r^\varepsilon\frac{\partial u_\varphi^\varepsilon}{\partial r} + \frac{u_\varphi^\varepsilon}{r}\frac{\partial u_\varphi^\varepsilon}{\partial\varphi} + u_z^\varepsilon\frac{\partial u_\varphi^\varepsilon}{\partial z} + \frac{u_r^\varepsilon u_\varphi^\varepsilon}{r} + \frac{1}{r}\frac{\partial p^\varepsilon}{\partial\varphi} = 0\,, \\
&-\mu\Delta u_z^\varepsilon + u_r^\varepsilon\frac{\partial u_z^\varepsilon}{\partial r} + \frac{u_\varphi^\varepsilon}{r}\frac{\partial u_z^\varepsilon}{\partial\varphi} + u_z^\varepsilon\frac{\partial u_z^\varepsilon}{\partial z} + \frac{\partial p^\varepsilon}{\partial z} = 0\,, \\
&\frac{\partial}{\partial r}(ru_r^\varepsilon) + \frac{\partial u_\varphi^\varepsilon}{\partial\varphi} + \frac{\partial}{\partial z}(ru_z^\varepsilon) = 0\,,
\end{aligned}
$$

where $u^\varepsilon = u_r^\varepsilon\vec{e}_r + u_\varphi^\varepsilon\vec{e}_\varphi + u_z^\varepsilon\vec{e}_z$ and $\Delta v = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v}{\partial\varphi^2} + \frac{\partial^2 v}{\partial z^2}$.

3.1. **Interior expansion.** Far from the ends of our cylinder $\mathcal{C}_\varepsilon$ i.e. for $z = 0, l$ we can neglect the local effects of the boundary conditions $u^\varepsilon(r,\varphi,z) = g_\alpha(\rho,\varphi)$, $\alpha = 0, l$ and try to find an ansatz that fits the system and the boundary condition on $r = R$, $R + \varepsilon h$. As in [3], [4], we seek an expansion in the form

$$
u^\varepsilon \sim u^0(\rho,\varphi,z) + \varepsilon u^1(\rho,\varphi,z) + \dots \tag{2}
$$

$$
p^\varepsilon \sim \frac{1}{\varepsilon^2}p^0(\rho,\varphi,z) + \frac{1}{\varepsilon}p^1(\rho,\varphi,z) + \dots \tag{3}
$$

where $\rho = \frac{r-R}{\varepsilon}$. Substituting these expansions into the Navier–Stokes equations and collecting equal powers of ε lead to the $\frac{1}{\varepsilon^2}$ term in the form

$$-\mu \frac{\partial^2 u_r^0}{\partial \rho^2} + \frac{\partial p^1}{\partial \rho} = 0\,,$$
$$-\mu \frac{\partial^2 u_\varphi^0}{\partial \rho^2} + \frac{1}{R}\frac{\partial p^0}{\partial \varphi} = 0\,,$$
$$-\mu \frac{\partial^2 u_z^0}{\partial \rho^2} + \frac{\partial p^0}{\partial z} = 0\,.$$

From the incompressibility equation we get

$$\frac{\partial u_r^0}{\partial \rho} = 0\,,$$
$$\frac{\partial u_\varphi^0}{\partial \varphi} + R\frac{\partial u_z^0}{\partial z} + u_r^0 = 0\,. \tag{4}$$

First we conclude that $u_r^0 = 0,\ p^1 = p^1(\varphi, z)$. Then we compute u_φ^0 and u_z^0 as

$$u_\varphi^0 = \frac{1}{2\mu R}(\rho - h)\rho\frac{\partial p^0}{\partial \varphi} + \omega(1 - \frac{\rho}{h})\,, \tag{5}$$

$$u_z^0 = \frac{1}{2\mu}(\rho - h)\rho\frac{\partial p^0}{\partial z}\ . \tag{6}$$

Equation (4) leads to

$$\frac{R}{2}(h-\rho)\rho\frac{\partial^2 p^0}{\partial z^2} + \frac{1}{2R}\frac{\partial}{\partial \varphi}[(h-\rho)\rho\frac{\partial p^0}{\partial \varphi}] = \frac{h'}{h^2}\rho\mu\omega\ .$$

Integrating with respect to ρ over $]0, h(\varphi)[$ and using a simple formula

$$\frac{\partial}{\partial \varphi}\int_0^{h(\varphi)} F(\rho,\varphi)d\rho = \int_0^{h(\varphi)} \frac{\partial F}{\partial \varphi}(\rho,\varphi)d\rho - F(h(\varphi),\varphi)h'(\varphi)$$

we get the Reynolds equation

$$Rh^3\frac{\partial^2 p^0}{\partial z^2} + \frac{1}{R}\frac{\partial}{\partial \varphi}(h^3\frac{\partial p^0}{\partial \varphi}) = 6h'\mu\omega \quad \text{in } \Omega =]0, 2\pi[\, \times\,]0, l[\,. \tag{7}$$

One boundary condition for p^0 is 2π–periodicity with respect to φ. Second boundary condition should be of the form

$$\frac{\partial p^0}{\partial z} = \lambda_0(\varphi) \ \text{ for } z = 0\,, \quad \frac{\partial p^0}{\partial z} = \lambda_l(\varphi) \ \text{ for } z = l\,. \tag{8}$$

The functions λ_α, $\alpha = 0, l$ are to be determined in the following section.

3.2. **Boundary layer.** On our interior expansion we did not impose any boundary condition at the ends $z = 0, l$. Therefore we, in general, have

$$u^0(\rho,\varphi,0) \neq g_0(\rho,\varphi)\,,$$
$$u^0(\rho,\varphi,l) \neq g_l(\rho,\varphi)\,.$$

We need to correct our expansion in the boundary layer near $z = 0$ and $z = l$. Near $z = 0$ we seek an expansion in the form

$$u^\varepsilon \sim u^0(\rho,\varphi,z) + B^0(\rho,\varphi,\xi) + \varepsilon[u^1(\rho,\varphi,z) + B^1(\rho,\varphi,\xi)] + \ldots$$
$$p^\varepsilon \sim \frac{1}{\varepsilon^2}p^0(\varphi,z) + \frac{1}{\varepsilon}[p^1(\varphi,z) + b^0(\rho,\varphi,\xi)] + p^2(\rho,\varphi,z) + b^1(\rho,\varphi,\xi) + \ldots$$

where $\rho = \frac{r-R}{\varepsilon}$ and $\xi = \frac{z}{\varepsilon}$ is the new dilated variable used to describe the fast changes of the solution in the boundary layer. Near $z = l$ we seek the expansion in the form

$$u^\varepsilon \sim u^0(\rho,\varphi,z) + H^0(\rho,\varphi,\tau) + \varepsilon[u^1(\rho,\varphi,z) + H^1(\rho,\varphi,\tau)] + \dots$$
$$p^\varepsilon \sim \frac{1}{\varepsilon^2} p^0(\varphi,z) + \frac{1}{\varepsilon}[p^1(\varphi,z) + h^0(\rho,\varphi,\tau)] + p^2(\rho,\varphi,z) + h^1(\rho,\varphi,\tau) + \dots$$

with $\tau = \frac{z-l}{\varepsilon}$. For the left boundary layer we get

$$\text{(9)} \qquad \begin{cases} -\mu\Delta_{\rho\xi} B_r^0 + \frac{\partial b^0}{\partial \rho} &= 0\,, \\ -\mu\Delta_{\rho\xi} B_z^0 + \frac{\partial b^0}{\partial \xi} &= 0\,, \\ -\mu\Delta_{\rho\xi} B_\varphi^0 &= 0\,, \end{cases}$$

in the infinite strip $\mathcal{G}(\varphi) =]0, h(\varphi)[\times]0, +\infty[$, where

$$\Delta_{\rho\xi} = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\xi^2}.$$

In addition we have

$$\text{(10)} \qquad \frac{\partial B_r^0}{\partial \rho} + \frac{\partial B_z^0}{\partial \xi} = 0\,.$$

The boundary conditions for B^0 are

$$\text{(11)} \qquad \begin{cases} B^0(\rho,\varphi,0) + u^0(\rho,\varphi,0) &= g_0(\rho,\varphi)\,, \\ B^0(0,\varphi,\xi) = B^0(h,\varphi,\xi) &= 0\,, \\ \lim_{\xi\to+\infty} B^0(\rho,\varphi,\xi) &= 0\,. \end{cases}$$

The *variable* φ is only a parameter. By integrating (10) over $\mathcal{G}(\varphi)$ we get the compatibility condition

$$0 = \int_0^{h(\varphi)} B_z^0(\rho,\varphi,0)d\rho = \int_0^{h(\varphi)} \vec{e}_z \cdot g_0(\rho,\varphi)d\rho - \lambda_0(\varphi)\int_0^{h(\varphi)} \frac{1}{2\mu}(\rho - h)\rho d\rho$$

leading to

$$\text{(12)} \qquad \lambda_0(\varphi) = -\frac{12\mu}{h^3(\varphi)} \int_0^{h(\varphi)} \vec{e}_z \cdot g_0(\rho,\varphi)d\rho.$$

The following result is well known and may be found for example in [8].

Lemma 1. *For every* $\varphi \in]0, 2\pi[$ *system* (9), (10), (11) *admits a unique solutions* $(B^0(\cdot,\varphi,\cdot), b^0(\cdot,\varphi,\cdot)) \in H^1(\mathcal{G}(\varphi))^3 \times L^2(\mathcal{G}(\varphi))/\mathbf{R}$.

Using solutions dependence on boundary condition $g_0 \in H^1(\mathcal{S}_1)$ and the geometry of the domain we get that $(B^0, b^0) \in H^1(\mathcal{G})^3 \times L^2(\mathcal{G})/\mathbf{R}$ where $\mathcal{G} = \{(\rho,\varphi,\xi); \varphi \in]0,2\pi[, (\rho,\xi) \in \mathcal{G}(\varphi)\}$. An analogous calculation gives the problem for (H^0, h^0):

$$\text{(13)} \qquad \begin{cases} -\mu\Delta_{\rho\tau} H_r^0 + \frac{\partial h^0}{\partial \rho} &= 0\,, \\ -\mu\Delta_{\rho\tau} H_z^0 + \frac{\partial h^0}{\partial \tau} &= 0\,, \\ \frac{\partial H_r^0}{\partial \rho} + \frac{\partial H_z^0}{\partial \tau} &= 0\,, \end{cases}$$

in $\mathcal{O}(\varphi) =]0, h(\varphi)[\times]-\infty, 0[$ where $\tau = \frac{z-l}{\varepsilon}$.

$$\left\{\begin{array}{rcl} H^0(\rho,\varphi,0) + u^0(\rho,\varphi,l) &=& g_l(\rho,\varphi)\,, \\ \lim_{\tau\to-\infty} H^0(\rho,\varphi,\tau) &=& 0\,, \\ -\mu\Delta_{\rho\tau} H^0_\varphi &=& 0\,, \\ H^0(0,\varphi,\tau) = H^0(h,\varphi,\tau) &=& 0\,, \end{array}\right. \tag{14}$$

which leads to

$$\lambda_l(\varphi) = -\frac{12\mu}{h^3(\varphi)} \int_0^{h(\varphi)} \vec{e}_z \cdot g_l(\rho,\varphi)d\rho\,. \tag{15}$$

For functions (B^0, b^0), (H^0, h^0) we have the Saint–Venant's principle:

Theorem 2. *There exist $C > 0$ and $\alpha > 0$ such that*

$$\begin{aligned} &|B^0|_{H^2(\{(\rho,\xi)\in\mathcal{G}(\varphi);\xi>t\})} < Ce^{-\alpha t}\,, \\ &|b^0|_{H^1(\{(\rho,\xi)\in\mathcal{G}(\varphi);\xi>t\})} < Ce^{-\alpha t}\,, \\ &|H^0|_{H^2(\{(\rho,\tau)\in\mathcal{O}(\varphi);\tau<-t\})} < Ce^{-\alpha t}\,, \\ &|h^0|_{H^1(\{(\rho,\tau)\in\mathcal{O}(\varphi);\tau<-t\})} < Ce^{-\alpha t}\,. \end{aligned}$$

For exponential decay of solutions of the Stokes equations see for example [8] and [4] and of the Laplace equation see for example [9].

Remark 1. In fact, it is easy to verify that

$$\begin{aligned} &B^0_\varphi(\rho,\varphi,\xi) \\ &= \sum_{k=1}^{+\infty} e^{-\frac{k\pi}{h(\varphi)}\xi} \left(\frac{2}{h(\varphi)} \int_0^{h(\varphi)} \sin\frac{k\pi\, t}{h(\varphi)}\, (g_0^\varphi(t,\varphi) - u^0_\varphi(t,\varphi,0))dt \right) \sin\frac{k\pi}{h(\varphi)}\rho \\ &= \sum_{k=1}^{+\infty} e^{-\frac{k\pi}{h(\varphi)}\xi} A_k(\varphi) \sin\frac{k\pi}{h(\varphi)}\rho \end{aligned} \tag{16}$$

leading to the asymptotic behaviour

$$B^0_\varphi \sim exp\left(-\frac{\pi}{\beta_1}\frac{z}{\varepsilon}\right)\,.$$

We get the analogous results for H^0_φ with the exponent $\frac{k\pi}{h(\varphi)}\tau$ instead of $-\frac{k\pi}{h(\varphi)}\xi$ and the coefficients $B_k(\varphi)$ analogous as $A_k(\varphi)$ but with g_l^φ instead of g_0^φ.

3.3. **Solvability of the Reynolds equation.** We can now write our Reynolds equation (governing the effective flow) in the form

$$\left\{\begin{array}{ll} Rh^3\frac{\partial^2 p^0}{\partial z^2} + \frac{1}{R}\frac{\partial}{\partial\varphi}(h^3\frac{\partial p^0}{\partial\varphi}) = 6h'\mu\omega & \text{in } \Omega\,, \\ \frac{\partial p^0}{\partial z} = -\frac{12\mu}{h^3(\varphi)}\int_0^{h(\varphi)} \vec{e}_z\cdot g_0(\rho,\varphi)d\rho & \text{for } z=0\,, \\ \frac{\partial p^0}{\partial z} = -\frac{12\mu}{h^3(\varphi)}\int_0^{h(\varphi)} \vec{e}_z\cdot g_l(\rho,\varphi)d\rho & \text{for } z=l\,, \\ p^0 \text{ is } 2\pi\text{–periodic in } \varphi\,. & \end{array}\right. \tag{17}$$

This is a Neumann's problem for linear elliptic equation and it has a unique (up to a constant) solution $p^0 \in H^3(\Omega) \cap C^2(\overline{\Omega})$ iff

$$\int_0^{2\pi}\int_0^{h(\varphi)} \vec{e}_z \cdot g_0(\rho,\varphi) d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi)} \vec{e}_z \cdot g_l(\rho,\varphi) d\rho d\varphi$$

which is exactly the hypothesis (**H3**).

The regularity of p^0 (due to the fact that $h' \in C^1(]0,2\pi[)$, $g_i \in C^2(\mathcal{S}_1)$) implies that $u^0 \in H^2(\mathcal{C}) \cap C^1(\overline{\mathcal{C}})$.

3.4. **Divergence Corrector.** We notice that (4) is not exactly satisfied. We only have that

$$\int_0^{h(\varphi)} \left(\frac{\partial u_\varphi^0}{\partial \varphi} + R\frac{\partial u_z^0}{\partial z}\right) d\rho = 0\,.$$

To fix that we add the divergence corrector in the form $\varepsilon\Psi(\rho,\varphi,z)\vec{e}_r$ with

$$\Psi(\rho,\varphi,z) = \int_0^{\rho} \left(\frac{\partial u_\varphi^0}{\partial \varphi} + R\frac{\partial u_z^0}{\partial z}\right)(t,\varphi,z)\, dt\,.$$

Now for the approximation

$$\begin{aligned} v^\varepsilon =& u^0((r-R)/\varepsilon,\varphi,z) - \varepsilon\Psi((r-R)/\varepsilon,\varphi,z)\vec{e}_r + \\ &+ H^0((r-R)/\varepsilon,\varphi,(z-l)/\varepsilon) + B^0((r-R)/\varepsilon,\varphi,z/\varepsilon) \end{aligned} \tag{18}$$

we get

$$\operatorname{div}\, v^\varepsilon = B_r^0 + H_r^0 + \frac{\partial H_\varphi^0}{\partial \varphi} + \frac{\partial B_\varphi^0}{\partial \varphi} + \varepsilon\Phi$$

where $|\Phi|_{L^\infty(\mathcal{C}_\varepsilon)} \le C$. By a simple change of variables we obtain

$$|B_r^0 + H_r^0 + \frac{\partial H_\varphi^0}{\partial \varphi} + \frac{\partial B_\varphi^0}{\partial \varphi}|_{L^r(\mathcal{C}_\varepsilon)} \le C\varepsilon^{2/r}\,, \quad 1 \le r < \infty\,. \tag{19}$$

4. Convergence

Our main result can be formulated as follows:

Theorem 3. *Let $(u^\varepsilon, p^\varepsilon)$ be the solution of the Navier-Stokes system* (1). *Let p^0 be the Reynolds pressure, i.e. the solution of the problem* (17) *and let u^0 be the Reynolds velocity given by* (5), (6). *Then*

$$\frac{1}{\sqrt{|\mathcal{C}_\varepsilon|}}|u^\varepsilon - u_\varepsilon^0|_{L^2(\mathcal{C}_\varepsilon)} \le C\sqrt{\varepsilon}\,,$$
$$\frac{1}{\sqrt{|\mathcal{C}_\varepsilon|}}|\varepsilon^2 p^\varepsilon - p^0|_{L^2(\mathcal{C}_\varepsilon)/\mathbf{R}} \le C\sqrt{\varepsilon}\ ,$$

where $u_\varepsilon^0(r,\varphi,z) = u^0((r-R)/\varepsilon,\varphi,z)$ and $|\mathcal{C}_\varepsilon| = \varepsilon\frac{l}{2}\int_0^{2\pi}(2h(\varphi)+\varepsilon h(\varphi)^2)d\varphi$.

Remark 2. The estimate in thin domain $\mathcal{C}_\varepsilon$ in the norm $|\ \ |_{L^2(\mathcal{C}_\varepsilon)}$ is worthless because the domain is shrinking. Convergence in such norm does not justify the lower-dimensional model since $|\phi|_{L^2(\mathcal{C}_\varepsilon)} \to 0$ for any bounded $\phi \in C(\mathbf{R}^3)$. The appropriate norm is $\|\phi\| = |\mathcal{C}_\varepsilon|^{-1/2}|\phi|_{L^2(\mathcal{C}_\varepsilon)}$. We notice that $\|1\| = 1 \not\to 0$ and that the convergence $\|\phi\| \to 0$ implies the convergence of the mean values

$$\frac{1}{\varepsilon}\int_R^{R+\varepsilon h} \phi \, dr \to 0 \quad \text{in} \quad L^2(\Omega) \ .$$

To prove Theorem 3 we need some technical results.

Lemma 2. (Poincare's inequality) *There exist a constant $C > 0$ such that*

$$|\phi|_{L^2(\mathcal{C}_\varepsilon)} \le C\varepsilon|\nabla\phi|_{L^2(\mathcal{C}_\varepsilon)} \tag{20}$$

for any $\phi \in H^1(\mathcal{C}_\varepsilon)$ such that $\phi = 0$ for $r = R + \varepsilon h$.

Proof. Let $\phi \in H^1(\mathcal{C}_\varepsilon)$ such that $\phi = 0$ for $r = R + \varepsilon h(\varphi)$. Using the cylindrical coordinates we have

$$\phi(r,\varphi,z) = \int_r^{R+\varepsilon h(\varphi)} \tfrac{\partial\phi}{\partial r}(t,\varphi,z)\, dt \, .$$

An easy application of the Schwartz inequality gives

$$\phi^2(r,\varphi,z) \le \left(\int_R^{R+\varepsilon h(\varphi)} (\tfrac{\partial\phi}{\partial r})^2(t,\varphi,z)t\, dt\right)\left(\int_r^{R+\varepsilon h(\varphi)} \frac{dt}{t}\right) \, .$$

Integration over $\mathcal{C}_\varepsilon$ leads to

$$\begin{aligned}
\int_{\mathcal{C}_\varepsilon} \phi^2 &= \int_0^{2\pi}\int_0^l\int_R^{R+\varepsilon h(\varphi)} \phi^2(r,\varphi,z) r dr dz d\varphi \\
&\le I(\varepsilon)\int_0^{2\pi}\int_0^l\int_R^{R+\varepsilon h(\varphi)} (\tfrac{\partial\phi}{\partial r})^2(t,\varphi,z)t\, dt\, dz\, d\varphi
\end{aligned}$$

where

$$I(\varepsilon) = \int_R^{R+\varepsilon\beta_2} r\int_r^{R+\varepsilon\beta_2}\frac{dt}{t}dr \le \frac{\beta_2^2}{2R}(R+\varepsilon\beta_2)\varepsilon^2 \, . \qquad \square$$

Lemma 3. *There exists $\phi \in H^1(\mathcal{C}_\varepsilon)$ such that*

$$\begin{cases}
div\ \phi = F & \in \ L^2(\mathcal{C}_\varepsilon) \\
\phi = \kappa\vec{e}_\varphi & \text{for} \ \ r = R, \ \kappa = \text{const.} \\
\phi = 0 & \text{for} \ \ r = R + \varepsilon h \\
\phi = \eta^\varepsilon & \text{for} \ \ z = 0\, , \ \ \eta^\varepsilon(r,\varphi) = \eta(\frac{r-R}{\varepsilon}, \varphi) \\
\phi = \delta^\varepsilon & \text{for} \ \ z = l\, , \ \ \delta^\varepsilon(r,\varphi) = \delta(\frac{r-R}{\varepsilon}, \varphi)\, ,
\end{cases}$$

where $\eta,\delta \in H^{1/2}(\mathcal{S}_1)$, $\mathcal{S}_1 = \{(\rho,\varphi); 0 < \rho < h(\varphi), \varphi \in]0,2\pi[\}$. $\eta,\delta = \kappa$ for $r = R$, $\eta,\delta = 0$ for $r = R + \varepsilon h$,

$$\int_R^{R+\varepsilon h}\int_0^{2\pi} r\vec{e}_z\cdot\eta^\varepsilon - \int_R^{R+\varepsilon h}\int_0^{2\pi} r\vec{e}_z\cdot\delta^\varepsilon = \int_0^l\int_R^{R+\varepsilon h}\int_0^{2\pi} rF\, ,$$

and

$$|\phi|_{H^1(\mathcal{C}_\varepsilon)} \le C\{\frac{1}{\varepsilon}|F|_{L^2(\mathcal{C}_\varepsilon)} + \frac{1}{\sqrt{\varepsilon}}(|\eta|_{H^{1/2}(\mathcal{S}_1)} + |\delta|_{H^{1/2}(\mathcal{S}_1)} + |\kappa|)\}\, .$$

If, in addition, $\eta, \delta \in W^{3/4,4}(\mathcal{S}_1)$ *we have the estimate*

$$|\phi|_{L^4(\mathcal{C}_\varepsilon)} \le C\{|F|_{L^4(\mathcal{C}_\varepsilon)} + \varepsilon^{1/4}(|\eta|_{W^{3/4,4}(\mathcal{S}_1)} + |\delta|_{W^{3/4,4}(\mathcal{S}_1)} + |\kappa|)\} \ .$$

Proof. Let $\phi = \phi^1 + \phi^2$ where ϕ^1 is a solution of the problem

$$\begin{cases} \text{div } \phi^1 = F & \text{in } \ \mathcal{C}_\varepsilon \,, \\ \phi^1 = \kappa \vec{e}_\varphi & \text{for } \ r = R \,, \\ \phi^1 = 0 & \text{for } \ r = R + \varepsilon h \,, \\ \phi^1 = (\eta^\varepsilon \cdot \vec{e}_\varphi)\vec{e}_\varphi + (\eta^\varepsilon \cdot \vec{e}_z)\vec{e}_z & \text{for } \ z = 0 \,, \\ \phi^1 = (\delta^\varepsilon \cdot \vec{e}_\varphi)\vec{e}_\varphi + (\delta^\varepsilon \cdot \vec{e}_z)\vec{e}_z & \text{for } \ z = l \,, \end{cases}$$

and ϕ^2 is a solution of the problem

$$\begin{cases} \text{div } \phi^2 = 0 & \text{in } \ \mathcal{C}_\varepsilon \,, \\ \phi^2 = 0 & \text{for } \ r = R, \ R + \varepsilon h \,, \\ \phi^2 = (\eta^\varepsilon \cdot \vec{e}_r)\vec{e}_r & \text{for } \ z = 0 \,, \\ \phi^2 = (\delta^\varepsilon \cdot \vec{e}_r)\vec{e}_r & \text{for } \ z = l \,. \end{cases}$$

We first deal with ϕ^1. We define Ψ^ε as a solution of the problem

$$\begin{cases} \frac{\partial \Psi^\varepsilon_r}{\partial \rho} + \frac{\partial \Psi^\varepsilon_\varphi}{\partial \varphi} + \frac{\partial \Psi^\varepsilon_z}{\partial z} = f_\varepsilon & \text{in } \mathcal{C} \,, \\ \Psi^\varepsilon = (0, \kappa, 0) & \text{for } \ \rho = 0 \,, \\ \Psi^\varepsilon = 0 & \text{for } \ \rho = h \,, \\ \Psi^\varepsilon = (0, \eta_\varphi, (R + \varepsilon\rho)\eta_z) & \text{for } \ z = 0 \,, \\ \Psi^\varepsilon = (0, \delta_\varphi, (R + \varepsilon\rho)\delta_z) & \text{for } \ z = l \,, \\ \Psi^\varepsilon \text{ is } 2\pi\text{–periodic in } \varphi \,, \end{cases}$$

where

$$\begin{aligned} &f_\varepsilon(\rho, \varphi, z) = (R + \varepsilon\rho)F(\Xi^{-1}(R + \varepsilon\rho, \varphi, z)) \,, \\ &\eta_\alpha(\rho, \varphi) = \eta^\varepsilon_\alpha(\Xi^{-1}(R + \varepsilon\rho, \varphi, 0)), \ \ \alpha = \varphi, z \,, \\ &\delta_\alpha(\rho, \varphi) = \delta^\varepsilon_\alpha(\Xi^{-1}(R + \varepsilon\rho, \varphi, l)), \ \ \alpha = \varphi, z \,, \\ &\Xi(x_1, x_2, x_3) = (r, \varphi, z) \,. \end{aligned}$$

The standard a priori estimate (see e.g.[8]) implies

$$|\Psi^\varepsilon|_{W^{1,q}(\mathcal{C})} \le C(|f_\varepsilon|_{L^q(\mathcal{C})} + |\eta|_{W^{1-1/q,q}(\mathcal{S}_1)} + |\delta|_{W^{1-1/q,q}(\mathcal{S}_1)} + |\kappa|),$$

where $L^q(\mathcal{C})$ and $W^{1,q}(\mathcal{C})$ are the usually defined spaces on $\mathcal{C} = \{(\rho, \varphi, z) \in \mathbf{R^3}; \varphi \in]0, 2\pi[, z \in]0, l[, 0 < \rho < h(\varphi)\}$ with respect to the Lebesque measure $d\rho d\varphi dz$. By direct integration we obtain

$$|f_\varepsilon|_{L^q(\mathcal{C})} \le \frac{C}{\varepsilon^{1/q}} |F|_{L^q(\mathcal{C}_\varepsilon)} \,.$$

Defining

$$\phi^1(x_1, x_2, x_3) = \frac{\varepsilon}{r} \Psi^\varepsilon_r(\frac{r - R}{\varepsilon}, \varphi, z)\vec{e}_r + \Psi^\varepsilon_\varphi(\frac{r - R}{\varepsilon}, \varphi, z)\vec{e}_\varphi + \frac{1}{r}\Psi^\varepsilon_z(\frac{r - R}{\varepsilon}, \varphi, z)\vec{e}_z \,,$$

we get estimates

$$|\phi^1|_{H^1(\mathcal{C}_\varepsilon)} \leq C(\frac{1}{\varepsilon}|F|_{L^2(\mathcal{C}_\varepsilon)} + \frac{1}{\sqrt{\varepsilon}}(|\eta|_{H^{1/2}(\mathcal{S}_1)} + |\delta|_{H^{1/2}(\mathcal{S}_1)} + |\kappa|))\,,$$

$$|\phi^1|_{L^4(\mathcal{C}_\varepsilon)} \leq C\varepsilon^{1/4}|\Psi^\varepsilon|_{L^4(\mathcal{C})} \leq C(|F|_{L^4(\mathcal{C}_\varepsilon)} + \varepsilon^{1/4}(|\eta|_{W^{3/4,4}(\mathcal{S}_1)} + |\delta|_{W^{3/4,4}(\mathcal{S}_1)} + |\kappa|))\,.$$

For ϕ^2 we proceed in a different way. We define the boundary layer–type functions χ^L and χ^R as the solutions of the problems

$$\begin{array}{llll}
\frac{\partial\chi_r^L}{\partial\rho} + \frac{\partial\chi_z^L}{\partial\xi} = 0 & \text{in } \mathcal{G}(\varphi)\,, & \frac{\partial\chi_r^R}{\partial\rho} + \frac{\partial\chi_z^R}{\partial\tau} = 0 & \text{in } \mathcal{O}(\varphi) \\
\chi^L = 0 & \text{for } \rho = 0, h(\varphi)\,, & \chi^R = 0 & \text{for } \rho = 0, h(\varphi)\,, \\
\chi^L = (\eta_r, 0) & \text{for } \xi = 0\,, & \chi^R = (\delta_r, 0) & \text{for } \tau = 0\,, \\
\chi^L \to 0 & \text{as } \xi \to +\infty\,, & \chi^R \to 0 & \text{as } \tau \to -\infty\,.
\end{array}$$

Those functions can be chosen such that they exponentially decay as $\xi \to +\infty$ and $\tau \to -\infty$ (see for example [8]). A simple change of variables gives for $\chi_\varepsilon^L = \chi^L(\frac{r-R}{\varepsilon}, \varphi, \frac{z}{\varepsilon})$

$$|\chi_\varepsilon^L|_{L^q(\mathcal{C}_\varepsilon)} \leq C\varepsilon^{\frac{2}{q}}|\chi^L|_{L^q(\omega)} \leq C\varepsilon^{\frac{2}{q}}|\eta|_{W^{1-1/q,q}(\mathcal{S}_1)}\,,$$

$$|\nabla\chi_\varepsilon^L|_{L^q(\mathcal{C}_\varepsilon)} \leq C\varepsilon^{\frac{2}{q}-1}|\nabla\chi^L|_{L^q(\omega)} \leq C\varepsilon^{\frac{2}{q}-1}|\eta|_{W^{1-1/q,q}(\mathcal{S}_1)}$$

with $\omega = \{(\rho,\varphi,\xi);\varphi \in]0,2\pi[, (\rho,\xi) \in \mathcal{G}(\varphi)\}$. We have similar estimates for function $\chi_\varepsilon^R = \chi^R(\frac{r-R}{\varepsilon}, \varphi, \frac{z-l}{\varepsilon})$. Now $\phi^2 = \chi_\varepsilon^L + \chi_\varepsilon^R + \vartheta$ where ϑ is defined by the problem

$$\begin{cases}
\operatorname{div}\vartheta = 0 & \text{in } \mathcal{C}_\varepsilon\,, \\
\vartheta = 0 & \text{for } r = R, R+\varepsilon h\,, \\
\vartheta = \chi_\varepsilon^L & \text{for } z = l\,, \\
\vartheta = \chi_\varepsilon^R & \text{for } z = 0\,,
\end{cases}$$

and having a $H^1(\mathcal{C}_\varepsilon)$ norm smaller than any power of ε (due to the Saint–Venant's principle). Now

$$|\phi^2|_{H^1(\mathcal{C}_\varepsilon)} \leq C(|\eta|_{H^{1/2}(\mathcal{S}_1)} + |\delta|_{H^{1/2}(\mathcal{S}_1)})\,,$$

$$|\phi^2|_{L^4(\mathcal{C}_\varepsilon)} \leq C\varepsilon^{1/2}(|\eta|_{W^{3/4,4}(\mathcal{S}_1)} + |\delta|_{W^{3/4,4}(\mathcal{S}_1)})\,.$$

□

For the solution of the problem (1) we prove the following estimates:

Proposition 1. *Let $(u^\varepsilon, p^\varepsilon)$ be the solution of the problem* (1). *Then there exist $N, C > 0$, independent on ε, such that*

$$\begin{aligned}
&|u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \leq N\sqrt{\varepsilon}\,, \\
&|\nabla u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \leq \frac{N}{\sqrt{\varepsilon}}\,, \qquad (21) \\
&|p^\varepsilon|_{L^2_0(\mathcal{C}_\varepsilon)} \leq \frac{C}{\varepsilon^{3/2}}\,.
\end{aligned}$$

Proof. Let ϕ_ε be the solution of the problem

$$\begin{cases} \text{div } \phi_\varepsilon = 0 & \text{in } \ \mathcal{C}_\varepsilon \,, \\ \phi_\varepsilon = \omega \vec{e}_\varphi & \text{for } \ r = R \,, \\ \phi_\varepsilon = 0 & \text{for } \ r = R + \varepsilon h \,, \\ \phi_\varepsilon = g_0(\frac{r-R}{\varepsilon}, \varphi) =: g_0^\varepsilon & \text{for } \ z = 0 \,, \\ \phi_\varepsilon = g_l(\frac{r-R}{\varepsilon}, \varphi) =: g_l^\varepsilon & \text{for } \ z = l \,. \end{cases}$$

Then from Lemma 3 we get estimates

$$|\phi_\varepsilon|_{H^1(\mathcal{C}_\varepsilon)} \le C \frac{1}{\sqrt{\varepsilon}} (|g_0|_{H^{1/2}(\mathcal{S}_\varepsilon)} + |g_l|_{H^{1/2}(\mathcal{S}_\varepsilon)} + \omega) \le \frac{C}{\sqrt{\varepsilon}} \,, \qquad |\phi_\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} \le C \varepsilon^{1/4} .$$

Multiplying the equation (1) by $u^\varepsilon - \phi_\varepsilon$ we get after integration over $\mathcal{C}_\varepsilon$

$$\mu |\nabla u^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} - \int_{\mathcal{C}_\varepsilon} (u^\varepsilon \nabla) \phi_\varepsilon u^\varepsilon + \int_{\mathcal{C}_\varepsilon} (u^\varepsilon \nabla) \phi_\varepsilon \phi_\varepsilon = \int_{\mathcal{C}_\varepsilon} \nabla u^\varepsilon \nabla \phi_\varepsilon \,.$$

It is easy to verify that the imbedding constant $H^1(\mathcal{C}_\varepsilon) \hookrightarrow L^4(\mathcal{C}_\varepsilon)$ can be chosen independently on ε. Then

$$|\int_{\mathcal{C}_\varepsilon} (u^\varepsilon \nabla) u^\varepsilon \phi_\varepsilon| \le |u^\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} |\nabla u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} |\phi_\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} \le C \varepsilon^{1/4} |\nabla u^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} \,,$$

$$|\int_{\mathcal{C}_\varepsilon} (u^\varepsilon \nabla) \phi_\varepsilon \phi_\varepsilon| \le |u^\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} |\nabla \phi_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} |\phi_\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} \le \varepsilon^{-1/4} C |\nabla u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \,.$$

Consequently we get

$$\mu |\nabla u^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} \le C(\varepsilon^{1/4} |\nabla u^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} + \frac{1}{\varepsilon^{1/4}} |\nabla u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} + \frac{1}{\sqrt{\varepsilon}} |\nabla u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)})$$

what leads to

$$|\nabla u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \le \frac{C}{\sqrt{\varepsilon}} \,.$$

To estimate the pressure, supposing that $\int_{\mathcal{C}_\varepsilon} p^\varepsilon = 0$, we define z_ε as the solution of the problem

$$\begin{cases} \text{div } z_\varepsilon = p^\varepsilon & \text{in } \ \mathcal{C}_\varepsilon \,, \\ z_\varepsilon = 0 & \text{on } \ \partial \mathcal{C}_\varepsilon \,. \end{cases}$$

Lemma 3 gives

$$|z_\varepsilon|_{H^1(\mathcal{C}_\varepsilon)} \le \frac{C}{\varepsilon} |p^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \,.$$

Using z_ε as the test function in (1) we obtain

$$\begin{aligned} |p^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} &= \mu \int_{\mathcal{C}_\varepsilon} \nabla u^\varepsilon \nabla z_\varepsilon - \int_{\mathcal{C}_\varepsilon} u^\varepsilon \nabla) u^\varepsilon z_\varepsilon \\ &\le \mu |\nabla u^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} |\nabla z_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} + C \varepsilon |\nabla u^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} |\nabla z_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \\ &\le \frac{C}{\varepsilon^{3/2}} |p^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \,. \end{aligned}$$

□

We define d^ε as a solution of the problem

$$\begin{cases} \text{div } d^\varepsilon = \text{div } \tilde{u}_\varepsilon & \text{in } \mathcal{C}_\varepsilon \, , \\ d^\varepsilon = \tilde{u}_\varepsilon & \text{on } \partial\mathcal{C}_\varepsilon \, , \end{cases}$$

where $\tilde{u}_\varepsilon = u^\varepsilon - v^\varepsilon$. Due to Lemma 3 d^ε can be chosen such that

$$|d^\varepsilon|_{H^1(\mathcal{C}_\varepsilon)} \leq C(\frac{1}{\varepsilon}|\text{div } v^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} + \frac{1}{\sqrt{\varepsilon}}(|\eta|_{H^{1/2}(\mathcal{S}_1)} + |\delta|_{H^{1/2}(\mathcal{S}_1)}))$$
$$|d^\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} \leq C(|\text{div } v^\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} + \varepsilon^{1/4}(|\eta|_{W^{3/4,4}(\mathcal{S}_1)} + |\delta|_{W^{3/4,4}(\mathcal{S}_1)}))$$

where $\eta(r,\varphi) = \varepsilon\Psi(\frac{r-R}{\varepsilon},\varphi,0)\vec{e}_r + H^0(\frac{r-R}{\varepsilon},\varphi,-\frac{l}{\varepsilon})$, $\delta(r,\varphi) = \varepsilon\Psi(\frac{r-R}{\varepsilon},\varphi,l)\vec{e}_r + B^0(\frac{r-R}{\varepsilon},\varphi,\frac{l}{\varepsilon})$. Using (19) we conclude that

$$|d^\varepsilon|_{H^1(\mathcal{C}_\varepsilon)} \leq C \quad , \quad |d^\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} \leq C\sqrt{\varepsilon} \ .$$

We denote by

$$\begin{cases} R^\varepsilon = u^\varepsilon - (v^\varepsilon + d^\varepsilon) \\ E^\varepsilon = p^\varepsilon - \frac{1}{\varepsilon^2}p^0 - \frac{1}{\varepsilon}(b^0 + h^0) + \frac{\partial\Psi}{\partial\rho} \, . \end{cases} \tag{22}$$

the difference between our approximation and the original solution, where v^ε is defined by (18). For $(R^\varepsilon, E^\varepsilon)$ we have the following estimates:

Proposition 2. *Let $(R^\varepsilon, E^\varepsilon)$ be defined by (22). Then there exists $C > 0$ independent from ε such that*

$$\frac{1}{\sqrt{|\mathcal{C}_\varepsilon|}}|R^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \leq C\sqrt{\varepsilon} \tag{23}$$

$$\frac{1}{\sqrt{|\mathcal{C}_\varepsilon|}}|\varepsilon^2 E^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)/\mathbf{R}} \leq C\sqrt{\varepsilon} \tag{24}$$

Proof. $(R^\varepsilon, E^\varepsilon)$ satisfy the system

$$\begin{cases} -\mu\Delta R^\varepsilon + ((v^\varepsilon + d^\varepsilon)\nabla)R^\varepsilon + (R^\varepsilon\nabla)u^\varepsilon + \nabla E^\varepsilon = \beta_\varepsilon \ \text{ in } \mathcal{C}_\varepsilon \, , \\ \text{div } R^\varepsilon = 0 \ \text{ in } \mathcal{C}_\varepsilon \, , \\ R^\varepsilon = 0 \ \text{ on } \partial\mathcal{C}_\varepsilon \, , \end{cases} \tag{25}$$

where, due to the regularity of u_0,

$$|\beta_\varepsilon|_{H^{-1}(\mathcal{C}_\varepsilon)} \leq C \, .$$

The explicit expression for β_ε is long and complicated but straightforward. Since it will not be used in the sequel it can be omitted. Multiplying the equation (25) by R^ε and integrating over $\mathcal{C}_\varepsilon$ we obtain

$$\begin{aligned}\mu|\nabla R^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} &= \int_{\mathcal{C}_\varepsilon} \beta_\varepsilon R^\varepsilon - \int_{\mathcal{C}_\varepsilon} (R^\varepsilon \nabla) u^\varepsilon R^\varepsilon \\ &= \int_{\mathcal{C}_\varepsilon} \beta_\varepsilon R^\varepsilon - \int_{\mathcal{C}_\varepsilon} (R^\varepsilon \nabla) v^\varepsilon R^\varepsilon + \int_{\mathcal{C}_\varepsilon} (R^\varepsilon \nabla) R^\varepsilon d^\varepsilon \\ &\leq C(|\nabla R^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} + |\nabla v^\varepsilon|_{L^\infty(\mathcal{C}_\varepsilon)} |R^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} \\ &\quad + |d^\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} |R^\varepsilon|_{L^4(\mathcal{C}_\varepsilon)} |\nabla R^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)}) \\ &\leq C(|\nabla R^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} + \frac{1}{\varepsilon}|R^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} + \sqrt{\varepsilon}|\nabla R^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)}) \\ &\leq C(|\nabla R^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} + \varepsilon|\nabla R^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)} + \sqrt{\varepsilon}|\nabla R^\varepsilon|^2_{L^2(\mathcal{C}_\varepsilon)}) \,.\end{aligned}$$

Thus $|\nabla R^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \leq C$. Using the Poincare's inequality we get

$$|R^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \leq C\varepsilon$$

what leads to (23). To estimate E^ε we define ϕ as the solution of the problem

$$\begin{cases} \operatorname{div} \phi = E^\varepsilon & \text{in } \ \mathcal{C}_\varepsilon \,, \\ \phi = 0 \text{ on } \ \partial\mathcal{C}_\varepsilon \,, \end{cases}$$

Lemma 3 gives that ϕ can be chosen such that

$$|\phi|_{H^1(\mathcal{C}_\varepsilon)} \leq \frac{C}{\varepsilon}|E^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)}.$$

Using ϕ as the test function in (25) we obtain

$$|E^\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \leq \frac{C}{\varepsilon}$$

and (24) easily follows. □

The above proposition proves Theorem 3.

The following estimates are direct consequences of the Proposition 2:

Corollary 1.

$$\left| \frac{1}{\varepsilon}\int_R^{R+\varepsilon h(\varphi)} u^\varepsilon \, dr + \frac{h^3}{12\mu}\left(\tfrac{\partial p^0}{\partial z}\vec{e}_z + \frac{1}{R}\tfrac{\partial p^0}{\partial \varphi}\vec{e}_\varphi\right) - \frac{\omega h}{2}\vec{e}_\varphi \right|_{L^2(\Omega)} \leq C\sqrt{\varepsilon} \,,$$

$$\left| \frac{\varepsilon}{h(\varphi)}\int_R^{R+\varepsilon h(\varphi)} p^\varepsilon \, dr - p^0 \right|_{L^2(\Omega)/\mathbf{R}} \leq C\sqrt{\varepsilon} \,.$$

Remark 3. If $g_0 = g_l = 0$ and $h =$const., then the only remaining term is $\frac{\omega h}{2}\vec{e}_\varphi$ i.e. there is only a uniform rotation of the fluid due to the rotation of the shaft with angular velocity ω.

Remark 4. Calculating in a similar way the second term u^1 in expansion (2) for u^ε, we get the Reynolds equation for the second term p^1 in expansion (3) for p.

$$\begin{cases} Rh^3\frac{\partial^2 p^1}{\partial z^2}+\frac{1}{R}\frac{\partial}{\partial\varphi}(h^3\frac{\partial p^1}{\partial\varphi})=-\frac{2\mu\omega}{R}hh'-\frac{h^4}{2}\frac{\partial^2 p^0}{\partial z^2}+\frac{1}{2R^2}\frac{\partial}{\partial\varphi}(h^4\frac{\partial p^0}{\partial\varphi}) & \text{in } \Omega\,, \\ \frac{\partial p^1}{\partial z}(\varphi,z)=\frac{12\mu}{h^3R}\,\tau_z(\varphi) \quad \text{for } \ z=0,l \\ p^1 \text{ is } 2\pi\text{–periodic in } \varphi\,, \end{cases}$$

where

$$\begin{aligned} \tau_0(\varphi)=\int_0^\infty\int_0^{h(\varphi)}(B_r^0+\frac{\partial B_\varphi^0}{\partial\varphi})d\rho d\xi &= -\int_0^{h(\varphi)}\rho\vec{e}_z\cdot g_0(\rho,\varphi)d\rho-\frac{h^4}{24\mu}\lambda_0(\varphi) \\ &\quad +2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\pi^2}\frac{\partial}{\partial\varphi}[h^2A_{2k+1}(\varphi)], \\ \tau_l(\varphi)=\int_0^\infty\int_0^{h(\varphi)}(H_r^0+\frac{\partial H_\varphi^0}{\partial\varphi})d\rho d\xi &= -\int_0^{h(\varphi)}\rho\vec{e}_z\cdot g_l(\rho,\varphi)d\rho-\frac{h^4}{24\mu}\lambda_l(\varphi) \\ &\quad +2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\pi^2}\frac{\partial}{\partial\varphi}[h^2B_{2k+1}(\varphi)]\,. \end{aligned}$$

Coefficients A_k are defined by

$$A_k(\varphi)=\int_0^{h(\varphi)}\sin\frac{k\pi\,t}{h(\varphi)}\ \left(g_0^\varphi(t,\varphi)-u_\varphi^0(t,\varphi)\right)dt\,,$$

i.e. those are the Fourier's coefficient for B_φ^0 (see (1)). Analogously

$$B_k(\varphi)=\int_0^{h(\varphi)}\sin\frac{k\pi\,t}{h(\varphi)}\ \left(g_l^\varphi(t,\varphi)-u_\varphi^0(t,\varphi)\right)dt$$

are the coefficients in the Fourier's expansion for H_φ^0.

This problem has a unique (up to a constant) solution $p^1\in H^2(\Omega)\cap C^1(\overline{\Omega})$ iff

$$\int_0^{2\pi}\int_0^{h(\varphi)}\rho\vec{e}_z\cdot g_0(\rho,\varphi)d\rho d\varphi=\int_0^{2\pi}\int_0^{h(\varphi)}\rho\vec{e}_z\cdot g_l(\rho,\varphi)d\rho d\varphi$$

which is exactly the hypothesis (**H2**). The error estimate of order $\varepsilon\sqrt{\varepsilon}$ can be proved analogously as in Theorem 3.

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