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NATURAL TRANSFORMATIONS OF SEPARATED JETS

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Abstract. Given a map of a product of two manifolds into a third one, one can define its jets of separated orders r and s. We study the functor $J^{r;s}$ of separated $(r;s)$-jets. We determine all natural transformations of $J^{r;s}$ into itself and we characterize the canonical exchange $J^{r;s} \to J^{s;r}$ from the naturality point of view.

Let M, N, Q be manifolds. Given a map $f : M \times N \to Q$, M. Kawaguchi introduced the concept of jet of separated orders r and s, [1], see also [5]. Write $J^{r;s}(M,N,Q)$ for the bundle of all such separated $(r;s)$-jets. In [2] the second author reformulated the Kawaguchi's idea in a way that clarifies there is a canonical exchange diffeomorphism $\varkappa_{M,N,Q} : J^{r;s}(M,N,Q) \to J^{s;r}(N,M,Q)$. Let $\mathcal{M}f$ be the category of all manifolds and all smooth maps and $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms. In Section 2 we interpret $J^{r;s}$ as a functor on the product category $\mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f$ similarly as the construction of classical r-jets is viewed as a functor on the category $\mathcal{M}f_m \times \mathcal{M}f$ in [3]. Then $\varkappa$ is a natural equivalence $J^{r;s} \to J^{s;r}$.

Our main problem is that of uniqueness of $\varkappa$ from the viewpoint of the theory of natural operations, [3]. In Proposition 4 we deduce that for $r \geq 2$, $s \geq 2$, $\varkappa$ is the only natural equivalence $J^{r;s} \to J^{s;r}$ over the canonical exchange functor $\mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f \to \mathcal{M}f_n \times \mathcal{M}f_m \times \mathcal{M}f$. For $r = 1$ or $s = 1$, the vector bundle structure of the classical first order jet bundles comes into play in a simple way. In order to prove Proposition 4, we determine all natural transformations $J^{r;s} \to J^{r;s}$ in Section 3. Here we use essentially a result from [4] that describes all natural transformations of the classical r-jet functor into itself.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [3].

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1. Separated $(r;s)$-jets

Consider three manifolds M, N, Q, two integers r, s and a point $(x,y) \in M \times N$. For every map $f : M \times N \to Q$, denote by $f_u : N \to Q$ or $f_v : M \to Q$ the partial map $v \mapsto f(u,v)$ or $u \mapsto f(u,v)$ respectively, $u \in M$, $v \in N$. If we construct the r-jet $j^r_x f_v$ for every $v \in N$, we obtain a map $N \to J^r_x(M,Q)$. Let $g : M \times N \to Q$ be another map.

Definition 1. We say that f and g determine the same jet of separated orders r and s at $(x,y) \in M \times N$, if

$$j^s_y(j^r_x f_v) = j^s_y(j^r_x g_v) \in J^r_y(N, J^r_x(M,Q)) . \tag{1}$$

The equivalence class will be denoted by $j^{r;s}_{x,y} f$. In short, $j^{r;s}_{x,y} f$ will be called the separated $(r;s)$-jet of f at (x,y).

Consider some local coordiates x^i on M, y^p on N and z^a on Q, $i = 1, \dots, m = \dim M$, $p = 1, \dots, n = \dim N$, $a = 1, \dots, q = \dim Q$. Write α or β for a multiindex corresponding to x^i or y^p, respectively. Let $f^a(x^i, y^p)$ be the coordinate expression of f. Since the coordinate form of $j^r_x f_v$ is determined by $D_\alpha f^a$, $0 \le |\alpha| \le r$, we have

Proposition 1. *$j^{r;s}_{x,y} f = j^{r;s}_{x,y} g$ is characterized by*

$$D_{\alpha\beta} f^a(x,y) = D_{\alpha\beta} g^a(x,y) , \qquad 0 \le |\alpha| \le r , \quad 0 \le |\beta| \le s . \tag{2}$$

Write $J^{r;s}(M,N,Q)$ for the space of all separated $(r;s)$-jets of $M \times N$ into Q. This is a fibered manifold over $M \times N \times Q$ with the induced coordinates

$$z^a_{\alpha\beta} , \qquad |\alpha| \le r , \quad |\beta| \le s . \tag{3}$$

Analogously to the classical case, $J^{r;s}_{x,y}(M,N,Q)_z \subset J^{r;s}(M,N,Q)$ means the subset of all separated $(r;s)$-jets with source (x,y) and target z, $x \in M$, $y \in N$, $z \in Q$.

For every $\overline{r} \le r$ and $\overline{s} \le s$, we have a canonical projection

$$\pi^{r,s}_{\overline{r},\overline{s}} : J^{r;s}(M,N,Q) \to J^{\overline{r};\overline{s}}(M,N,Q) .$$

Write $\varepsilon : M \times N \to N \times M$ for the exchange map $\varepsilon(x,y) = (y,x)$. Using (2) we find that $j^{s;r}_{y,x}(f \circ \varepsilon)$ is determined by $j^{r;s}_{x,y} f$. This defines a canonical exchange diffeomorphism

$$\varkappa_{M,N,Q} : J^{r;s}(M,N,Q) \to J^{s;r}(N,M,Q) . \tag{4}$$

Example 1. For $M = N = \mathbb{R}$, $x = y = 0$, $r = s = 1$ we have $J^1_0(\mathbb{R}, J^1_0(\mathbb{R}, Q)) = T(TQ)$. In this case, the restriction of $\varkappa_{\mathbb{R},\mathbb{R},Q}$ coincides with the well known canonical involution on TTQ.

2. The functor $J^{r;s}$

Consider another manifold $\overline{Q}$.

Lemma 1. *Let $g : Q \to \overline{Q}$ be a map and $X = j^{r;s}_{x,y} f \in J^{r;s}(M,N,Q)$. Then $j^{r;s}_{x,y}(g \circ f) \in J^{r,s}(M,N,\overline{Q})$ depends on $j^{r+s}_{f(x,y)} g$ and X only.*

Proof. In coordinates, the derivatives in question of $g \circ f$ depend on the derivatives of g up to order $r+s$ and on X only. □

Thus, for every $W \in J^{r+s}_z(Q,\overline{Q})_w$ and every $X \in J^{r;s}_{x,y}(M,N,Q)_z$, we have defined a composition

$$W \circ X \in J^{r;s}_{x,y}(M,N,\overline{Q})_w\,. \tag{5}$$

In the same way, we deduce

Lemma 2. *Let $g : \overline{M} \to M$ and $h : \overline{N} \to N$ be two maps, $g(\overline{x}) = x$, $h(\overline{y}) = y$, $\overline{x} \in \overline{M}$, $\overline{y} \in \overline{N}$ and $X = j^{r;s}_{x,y} f \in J^{r;s}(M,N,Q)$. Then $j^{r;s}_{\overline{x},\overline{y}}(f \circ (g \times h)) \in J^{r;s}_{\overline{x},\overline{y}}(\overline{M},\overline{N},Q)$ depends on $j^r_{\overline{x}} g$, $j^s_{\overline{y}} h$ and X only.* □

Thus, for $Y \in J^r_{\overline{x}}(\overline{M},M)_x$, $Z \in J^s_{\overline{y}}(\overline{N},N)_y$ and $X \in J^{r;s}_{x,y}(M,N,Q)_z$ we have defined the composition

$$X \circ (Y,Z) \in J^{r;s}_{\overline{x},\overline{y}}(\overline{M},\overline{N},Q)_z\,. \tag{6}$$

If we combine both (5) and (6), we obtain

$$W \circ X \circ (Y,Z) \in J^{r;s}_{\overline{x},\overline{y}}(\overline{M},\overline{N},\overline{Q})_w\,. \tag{7}$$

The associativity properties of (7) follow directly from the associativity of the composition of maps.

Consider two local diffeomorphisms $g : M \to \overline{M}$, $h : N \to \overline{N}$ and a map $f : Q \to \overline{Q}$. Then we define

$$J^{r;s}(g,h,f) : J^{r;s}(M,N,Q) \to J^{r;s}(\overline{M},\overline{N},\overline{Q}) \tag{8}$$

by setting, for every $X \in J^{r;s}_{x,y}(M,N,Q)_z$, $g(x) = \overline{x}$, $h(y) = \overline{y}$,

$$J^{r;s}(g,h,f)(X) = (j^{r+s}_z f) \circ X \circ ((j^r_{\overline{x}} g^{-1}, j^s_{\overline{y}} h^{-1}))\,, \tag{9}$$

where g^{-1} and h^{-1} are constructed locally.

Clearly, using the terminology of [3], we obtain

Proposition 2. *$J^{r;s}$ is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f$.* □

Remark 1. It is interesting to discuss the order of $J^{r;s}$. In general, a bundle functor F on the product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ of k categories over manifolds will be called of order $(r_1,\dots,r_k)$, if for every two k-tuples of $\mathcal{C}_i$-morphisms f_i, $g_i : A_i \to B_i$, $i = 1,\dots,k$, the conditions $j^{r_i}_{x_i} f_i = j^{r_i}_{x_i} g_i$, $x_i \in A_i$, imply

$$F(f_1,\dots,f_k)|F_{x_1,\dots,x_k}(A_1,\dots A_k) = F(g_1,\dots,g_k)|F_{x_1,\dots,x_k}(A_1,\dots,A_k)\,. \tag{10}$$

In our case, the order of $J^{r;s}$ is $(r,s,r+s)$.

3. Natural transformations $J^{r;s} \to J^{r;s}$

In the case of the classical r-jet functor J^r, which is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, the following list of all natural transformations $J^r \to J^r$ is deduced in [4]. For a map $f : M \to N$, let f^0_x, $x \in M$, denote the constant map $f^0_x(u) = x$. The so-called contraction $\sigma_{M,N} : J^r(M,N) \to J^r(M,N)$ is defined by

$$\sigma_{M,N}(j^r_x f) = j^r_x(f^0_x)\,.$$

For $r \geq 2$, all natural transformations $J^r \to J^r$ are

$$\mathrm{id}_{J^r(M,N)} \quad \text{and} \quad \sigma_{M,N}\,. \tag{11}$$

For $r = 1$, $J^1(M,N) = T^*M \otimes TN$ is a vector bundle and all natural transformations $J^1 \to J^1$ are the homotheties

$$k\,\mathrm{id}_{J^1(M,N)}\,, \qquad k \in \mathbb{R}\,. \tag{12}$$

Having a map $f : M \times N \to Q$, we define $f^i_{x,y} : M \times N \to Q$, $x \in M$, $y \in N$, $i = 0, 1, 2$, by

$$f^0_{x,y}(u,v) = f(x,y)\,, \quad f^1_{x,y}(u,v) = f(x,v)\,, \quad f^2_{x,y}(u,v) = f(u,y)\,.$$

Then we introduce the following three natural transformations
$\varrho^i_{M,N,Q} : J^{r;s}(M,N,Q) \to J^{r;s}(M,N,Q)$

$$\varrho^0_{M,N,Q}(j^{r;s}_{x,y} f) = j^{r;s}_{x,y} f^0_{x,y} \qquad \text{(the total contraction)}, \tag{13}$$

$$\varrho^1_{M,N,Q}(j^{r;s}_{x,y} f) = j^{r;s}_{x,y} f^1_{x,y} \qquad \text{(the first contraction)}, \tag{14}$$

$$\varrho^2_{M,N,Q}(j^{r;s}_{x,y} f) = j^{r;s}_{x,y} f^2_{x,y} \qquad \text{(the second contraction)}. \tag{15}$$

For $s = 1$ (the case $r = 1$ is quite similar), we can construct further natural transformations as follows. We recall

$$J^{r;1}(M,N,Q) = \bigcup_{x \in M} J^1(N, J^r_x(M,Q))\,.$$

Take any natural transformation $\tau_{M,Q} : J^r(M,Q) \to J^r(M,Q)$, see (11) or (12). Consider the restriction

$$(\tau_{M,Q})_x : J^r_x(M,Q) \to J^r_x(M,Q)\,, \qquad x \in M\,,$$

and construct the induced jet map

$$J^1(\mathrm{id}_N, (\tau_{M,Q})_x) : J^1(N, J^r_x(M,Q)) \to J^1(N, J^r_x(M,Q))\,.$$

Taking into account all $x \in M$, we obtain a map

$$\mathcal{J}^1_N \tau_{M,Q} : J^{r;1}(M,N,Q) \to J^{r;1}(M,N,Q)\,.$$

Applying further a homothety with coefficient $k \in \mathbb{R}$ on each vector bundle $J^1(N, J^r_x(M,Q))$, we obtain a natural transformation

$$k\mathcal{J}^1_N \tau_{M,Q} : J^{r;1}(M,N,Q) \to J^{r;1}(M,N,Q)\,.$$

For $r \geq 2$, the only two possibilities are (11). For $r = 1$, we have $\tau_{M,Q} = \overline{k}\,\mathrm{id}_{J^1(M,Q)}$, $\overline{k} \in \mathbb{R}$.

From the technical point of view, our main result is the following assertion.

Proposition 3. *All natural transformations $J^{r;s} \to J^{r;s}$ are*
(i) *for $r \geq 2$, $s \geq 2$*

$$\varrho^0, \varrho^1, \varrho^2, \mathrm{id}\,, \tag{16}$$

(ii) *for $s = 1$, $r \geq 2$ (and analogously for $r = 1, s \geq 2$)*

$$k\mathcal{J}^1\sigma\,,\ k\mathcal{J}^1\,\mathrm{id}\,, \qquad k \in \mathbb{R}\,, \tag{17}$$

(iii) *for $r = 1$, $s = 1$*

$$k\mathcal{J}^1\overline{k}\,\mathrm{id}\,, \qquad k, \overline{k} \in \mathbb{R}\,. \tag{18}$$

Proof. First of all we discuss the subcategory $\mathcal{M}f_q \subset \mathcal{M}f$. Applying Lemma 14.11 from [3] to each factor of $\mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f_q$, we deduce that every natural transformation of $J^{r;s}$ into itself is over the identities on bases. Write $G = G^r_m \times G^s_n \times G^{r+s}_q$ and $L^{r;s}_{m,n,q} = J^{r;s}_{0,0}(\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^q)_0$. According to the general theory, [3], we are looking for G-equivariant maps of $L^{r;s}_{m,n,q}$ into itself. By (3), the canonical coordinates on $L^{r;s}_{m,n,q}$ are

$$z^a_{\alpha\beta}\,, \quad |\alpha| \leq r\,,\ |\beta| \leq s\,, \quad (\alpha, \beta) \neq (0,0)\,. \tag{19}$$

The action of $G^1_m \times G^1_n \times G^1_q \subset G$ on (19) is tensorial.

Any smooth map $f : L^{r;s}_{m,n,q} \to L^{r;s}_{m,n,q}$ is of the form

$$\overline{z}^a_{\alpha\beta} = f^a_{\alpha\beta}(z^b_{\gamma\delta})\,, \tag{20}$$

where γ or δ is a multiindex corresponding to x^i or y^p, respectively, and $b = 1, \dots, q$. By the homogeneous function theorem, [3], p. 213, the homotheties in G^1_q yield $f^a_{\alpha\beta}$ is linear in $z^b_{\gamma\delta}$. Then the homotheties in G^1_m and G^1_n imply that $f^a_{\alpha\beta}$ depends on $z^b_{\gamma\delta}$ with $|\alpha| = |\gamma|$, $|\beta| = |\delta|$ only. Using the generalized invariant tensor theorem, [3], p. 230, we obtain

$$\overline{z}^a_{\alpha\beta} = k_{|\alpha|,|\beta|} z^a_{\alpha\beta}\,, \qquad k_{|\alpha|,|\beta|} \in \mathbb{R}\,. \tag{21}$$

Now we proceed by induction with respect to $r + s$. For $r + s = 1$, (21) reads

$$\overline{z}^a_i = k_{1,0} z^a_i\,, \qquad \overline{z}^a_p = k_{0,1} z^a_p\,. \tag{22}$$

Consider the kernel K of the jet projection $G^{r+s}_q \to G^{r+s-1}_q$ together with the units of G^r_m and G^s_n. Hence the canonical coordinates on K are

$$A^a_{b_1 \dots b_{r+s}}$$

symmetric in all subscripts. Since the action of G on $L^{r;s}_{m,n,q}$ is given by the jet composition, we have, provided we write explicitely $\alpha = (i_1, \dots, i_r)$, $\beta = (p_1, \dots, p_s)$,

$$\overline{z}^a_{i_1\dots i_r p_1\dots p_s} = z^a_{i_1\dots i_r p_1\dots p_s} + A^a_{b_1\dots b_r b_{r+1}\dots b_{r+s}} z^{b_1}_{i_1} \dots z^{b_r}_{i_r} z^{b_{r+1}}_{p_1} \dots z^{b_{r+s}}_{p_s}\,, \tag{23}$$

while the other coordinates on $L^{r;s}_{m,n,q}$ are unchanged. The equivariancy of (21) with $|\alpha| = r$, $|\beta| = s$ with respect to (23) reads

$$\begin{aligned} &k_{r,s} z^a_{i_1\dots p_s} + k^r_{1,0} k^s_{0,1} A^a_{b_1\dots b_{r+s}} z^{b_1}_{i_1} \dots z^{b_{r+s}}_{p_s} = \\ &\quad = k_{r,s}(z^a_{i_1\dots p_s} + A^a_{b_1\dots b_{r+s}} z^{b_1}_{i_1} \dots z^{b_{r+s}}_{p_s})\,. \end{aligned} \tag{24}$$

This implies

$$k_{r,s} = k^r_{1,0} k^s_{0,1}\,. \tag{25}$$

The action of G on the subspace (z^a_α) or (z^a_β), i.e. $|\beta| = 0$ or $|\alpha| = 0$, respectively, corresponds to the classical jet case. Thus, for $r \geq 2$, $s \geq 2$, (11) yields the following four possibilities

$$k_{1,0} = 0, 1\,, \qquad k_{0,1} = 0, 1\,. \tag{26}$$

Then (25) leads to the coordinate form of the four possibilities of (i). For $r \geq 2$ and $s = 1$, (11) and (12) yield $k_{1,0} = 0, 1$, $k_{0,1} = k \in \mathbb{R}$. Then (25) implies (ii). For $r = 1$ and $s = 1$, (12) yields $k_{1,0} = k$, $k_{0,1} = \overline{k}$. Then (25) implies (iii).

To extend our result from the subcategory $\mathcal{M}f_q$ to the whole category $\mathcal{M}f$, it suffices to consider naturality with respect to the canonical injections $\mathbb{R}^q \to \mathbb{R}^{q+1}$ for all q. □

4. The uniqueness of $\varkappa$

In general, consider three categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and a functor $\varphi : \mathcal{C} \to \mathcal{D}$. A natural transformation over φ of two functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ means a natural transformation $F \to G \circ \varphi$.

In our case, $J^{r;s}$ is a functor on $\mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f$. Denote by E the exchange functor $E : \mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f \to \mathcal{M}f_n \times \mathcal{M}f_m \times \mathcal{M}f$, $E(M, N, Q) = (N, M, Q)$, $E(g, h, f) = (h, g, f)$. Then the canonical exchange $\varkappa : J^{r;s} \to J^{s;r}$, see (4), is a natural equivalence over E.

Let $\tau : J^{r;s} \to J^{s;r}$ be a natural transformation over E. Then $\varkappa^{-1} \circ \tau$ is a natural transformation $J^{r;s} \to J^{r;s}$ over the identity of $\mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f$. These are listed in Proposition 3. Thus, we have deduced

Proposition 4. *All natural transformations $J^{r;s} \to J^{s;r}$ over E are*

(i) $\varkappa$, $\varkappa \circ \varrho^0$, $\varkappa \circ \varrho^1$, $\varkappa \circ \varrho^2$ *for* $r \geq 2$, $s \geq 2$,

(ii) $\varkappa \circ k\mathcal{J}^1\sigma$, $\varkappa \circ k\mathcal{J}^1 \mathrm{id}$, $k \in \mathbb{R}$ *for* $r \geq 2$, $s = 1$,

(iii) $\varkappa \circ k\mathcal{J}^1\overline{k}\,\mathrm{id}$, $k, \overline{k} \in \mathbb{R}$ *for* $r = 1$, $s = 1$.

In particular, for $r \geq 2$, $s \geq 2$, $\varkappa$ is the only natural equivalence $J^{r;s} \to J^{s;r}$ over E. □

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