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ITO EQUATION AS A GEODESIC FLOW ON $\widehat{Diff^s(S^1)\bigodot C^\infty}(S^1)$

PARTHA GUHA

Abstract. The Ito equation is shown to be a geodesic flow of L^2 metric on the semidirect product space $\widehat{Diff^s(S^1)\bigodot C^\infty}(S^1)$, where $Diff^s(S^1)$ is the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. We also study a geodesic flow of a H^1 metric.

1. Introduction

It is known that the periodic Korteweg-de Vries (KdV) equation can be interpreted as geodesic flow of the right invariant metric on the Bott-Virasoro group, which at the identity is given by the L^2- inner product [15,18].

Recently Misiolek [14] and others [7,12,17] showed that an analogous correspondence can be established for the Camassa-Holm equation [4]. It gives rise to a geodesic flow of a certain right invariant Sobolev metric H^1 on the Bott-Virasoro group.

Thus we see the KdV and the Camassa-Holm equations arise in a unified geometric construction, both are integrable systems which describe geodesic flows on the Bott-Virasoro group. Earlier it was known that both the KdV and the Camassa-Holm are obtained from different regularisations of the Euler equation for a one dimensional compressible fluid. The Euler equation, of course, describes geodesic motion on the group of orientation preserving diffeomorphisms of the circle $Diff(S^1)$ with respect to L^2 metric [6].

Following Ebin-Marsden [6] we enlarge $Diff(S^1)$ to a Hilbert manifold $Diff^s(S^1)$, the diffeomorphism of Sobolev class H^s. This is a topological space. If $s > n/2$, it makes sense to talk about an H^s map from one manifold to another. Using local charts, one can check whether the derivation of order $\leq s$ are square integrable.

The Lie algebra of $\widehat{Diff^s(S^1)\bigodot C^\infty}(S^1)$ has a three dimensional extension (explained in the next section)

$$Vect^s(S^1)\bigodot C^\infty(S^1)\oplus \mathbf{R}^3\,.$$

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Then a typical element of this algebra would be

$$(f\frac{d}{dx}, u(x), \alpha) \quad \text{where } f\frac{d}{dx} \in \mathit{Vect}(S^1), \quad u(x) \in C^\infty(S^1) \quad \alpha \in \mathbf{R}^3 .$$

The $\mathit{Diff}^s\widehat{(S^1)\bigodot}C^\infty(S^1)$ is the non-trivial extension of $\mathit{Diff}^s(S^1)\bigodot C^\infty(S^1)$.

In this paper we study a geodesic flow on the $\mathit{Diff}^s\widehat{(S^1)\bigodot}C^\infty(S^1)$, which at the identity is given by the L^2 inner product, is a completely integrable coupled nonlinear third order partial differential equation introduced by M. Ito [9]. Hence the Ito equation arises as a geodesic flow, in general of course, these flows are not integrable.

Then we study a geodesic flow of the right invariant inner metric on the $\mathit{Diff}^s\widehat{(S^1)\bigodot}C^\infty(S^1)$, which at the identity is given by the H^1 inner product. Thus we obtain a new coupled nonlinear integrable system. The relation between the Ito and this new system is the same as the relation between the KdV and the Camassa-Holm equations.

Now we state our main result:

Theorem 1. *Let $t \mapsto \hat{c}$ be a curve in the $\mathit{Diff}^s\widehat{(S^1)\bigodot}C^\infty(S^1)$. Let $\hat{c} = (e, e, 0)$ be the initial point, directing to the vector $\hat{c}(0) = (u(x)\frac{d}{dx}, v(x), \gamma_0)$, where $\gamma_0 \in \mathbf{R}^3$. Then $\hat{c}(t)$ is a geodesic of the*
(A) *L^2 metric if and only if $(u(x,t)\frac{d}{dx}, v(x,t), \gamma)$ satisfies the Ito equation*

$$u_t + u_{xxx} + 6uu_x + 2vv_x = 0 ,$$
$$v_t + 2(uv)_x = 0 ,$$
$$\gamma_t = 0 .$$

(B) *H^1 metric if and only if $(u(x,t)\frac{d}{dx}, v(x,t), \gamma)$ satisfies*

$$u_t - u_{xxt} + uu_{xxx} + 2u_x u_{xx} - u_{xxx} + v_x v_{xx} - 5uu_x - vv_x = 0 ,$$
$$v_t - v_{xxt} + uv_{xxx} - uv_x + u_x v_{xx} - u_x v = 0 .$$

The Ito system [9] admits a bi-Hamiltonian structure

$$\mathbf{D}_2 \delta H_n = \mathbf{D}_1 \delta H_{n+1} ,$$

where

$$\mathbf{D}_2 = \begin{pmatrix} D^3 + 4uD + 2u_x & 2vD \\ 2v_x + 2vD & 0 \end{pmatrix}$$

$$\mathbf{D}_1 = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$$

with the Hamiltonian functionals

$$H_1[u,v] = \frac{1}{2}\int (u^2 + v^2)dx \tag{1}$$

$$H_2[u,v] = \frac{1}{2}\int (u^3 - \frac{1}{2}{u_x}^2 + uv^2)dx . \tag{2}$$

The recursion operator arising from a Hamiltonian pair

$$\mathbf{R} = \mathbf{D}_2{\mathbf{D}_1}^{-1} = \begin{pmatrix} D^2 + 4u + 2u_x D^{-1} & 2v \\ 2v_x D^{-1} + 2v & 0 \end{pmatrix}$$

is a hereditory operator yields infinitely many conserved quantities [1].

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2. Ito equation and L^2 metric

Let $Diff^s(S^1)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. It is known that the group $Diff^s(S^1)$ as well as its Lie algebra of vector fields on S^1, $T_{id}Diff^s(S^1) = Vect^s(S^1)$, have non-trivial one-dimensional central extensions, the Bott-Virasoro group $\widehat{Diff}^s(S^1)$ and the Virasoro algebra Vir respectively [10,11,18].

The Lie algebra $Vect^s(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$[f\frac{d}{dx}, g\frac{d}{dx}] := (f(x)g'(x) - f'(x)g(x))\frac{d}{dx}\,. \tag{3}$$

One parameter family of $Vect^s(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ by

$$L^{(\mu)}_{f(x)\frac{d}{dx}} a(x) = f(x)a'(x) - \mu f'(x)a(x)\,, \tag{4}$$

where

$$L^{(\mu)}_{f(x)\frac{d}{dx}} = f(x)\frac{d}{dx} - \mu f'(x)$$

is the derivative with respect to the vector field $f(x)\frac{d}{dx}$.

The Lie algebra of $Diff^s(S^1) \bigodot C^\infty(S^1)$ is the semidirect product Lie algebra

$$\mathcal{G} = Vect^s(S^1) \bigodot C^\infty(S^1)\,.$$

An element of $\mathcal{G}$ is a pair $(f(x)\frac{d}{dx}, a(x))$, where $f(x)\frac{d}{dx} \in Vect^s(S^1)$ and $a(x) \in C^\infty(S^1)$.

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles

$$\omega_1((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f'(x)g''(x)dx \tag{5}$$

$$\omega_2((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = \int_{S^1} f''(x)b(x) - g''a(x))dx \tag{6}$$

$$\omega_3((f\frac{d}{dx}, a), (g\frac{d}{dx}, b)) = 2\int_{S^1} a(x)b'(x)dx\,. \tag{7}$$

The first cocycle ω_1 is the well known Gelfand-Fuchs cocycle. The Virasoro algebra is the unique non-trivial central extension of $Vect(S^1)$ via this ω_1 cocycle. Hence we define the Virasoro algebra

$$Vir = Vect^s(S^1) \oplus \mathbf{R}.$$

The space $C^\infty(S^1) \oplus \mathbf{R}$ is identified with a part of the dual space to the Virasoro algebra. It is called the *regular part*, and the pairing between this space and the Virasoro algebra is given by:

$$\langle (u(x), a), (f(x)\frac{d}{dx}, \alpha) \rangle = \int_{S^1} u(x) f(x) dx + a\alpha.$$

Similarly we consider an extension of $\mathcal{G}$. This extended algebra is given by

$$\hat{\mathcal{G}} = Vect^s(S^1) \bigodot C^\infty(S^1) \oplus \mathbf{R}^3. \tag{8}$$

The Lie algebra $\hat{\mathcal{G}}$ has been considered in various places [2,8,13]. It was shown in [16] that the cocycles define the universal central extension the Lie algebra $Vect^s(S^1) \bigodot C^\infty(S^1)$. This means $H^2(Vect(S^1) \bigodot C^\infty(S^1)) = \mathbf{R}^3$.

Definition 1. The commutation relation in $\hat{\mathcal{G}}$ is given by

$$[(f\frac{d}{dx}, a, \alpha), (g\frac{d}{dx}, b, \beta)] := ((fg' - f'g)\frac{d}{dx}, fb' - ga', \omega) \tag{9}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbf{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ are the two cocycles.

The dual space of smooth functions $C^\infty(S^1)$ is the space of distributions (generalized functions) on S^1. Of particular interest are the orbits in $\hat{\mathcal{G}}^*_{reg}$. In the case of current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations.

Definition 2. The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ as follows: Consider

$$\hat{\mathcal{G}}^*_{reg} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbf{R}^3.$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $\langle \cdot, \cdot \rangle : \hat{\mathcal{G}}^*_{reg} \otimes \hat{\mathcal{G}} \to \mathbf{R}$:

$$\langle \hat{u}, \hat{f} \rangle = \int_{S^1} f(x) u(x) dx + \int_{S^1} a(x) v(x) dx + \alpha \cdot \gamma, \tag{10}$$

where $\hat{u} = (u(x), v, \gamma)$, $\hat{f} = (f\frac{d}{dx}, a, \alpha)$.

Extend (8) to a right invariant metric on the semi-direct product space $\widehat{Diff^s(S^1) \bigodot C^\infty(S^1)}$ by setting

$$\langle \hat{u}, \hat{f} \rangle_{\hat{\xi}} = \langle d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{u}, d_{\hat{\xi}} R_{\hat{\xi}^{-1}} \hat{f} \rangle_{L^2} \tag{11}$$

for any $\hat{\xi} \in \hat{\mathcal{G}}$ and $\hat{u}, \hat{f} \in T_{\hat{\xi}}\widehat{\mathcal{G}}$, where

$$R_{\hat{\xi}} : \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$$

is the right translation by $\hat{\xi}$.

We shall show that the Ito equation is precisely the Euler-Arnold equation on the dual space of $\hat{\mathcal{G}}$ associated with the L^2 inner product.

Given any three elements

$$\hat{f} = (f\frac{d}{dx}, a, \alpha), \quad \hat{g} = (g\frac{d}{dx}, b, \beta) \ \hat{u} = (u\frac{d}{dx}, v, \gamma)$$

in $\hat{\mathcal{G}}$.

Lemma 1.

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1 f''' + c_2 a'' \\ f'v(x) + f(x)v'(x) - c_2 f'' + 2c_3 a'(x) \\ 0 \end{pmatrix}$$

Proof. This follows from

$$\begin{aligned}\langle ad^*_{\hat{f}}\hat{u}, \hat{g}\rangle_{L^2} =& \langle \hat{u}, [\hat{f}, \hat{g}]\rangle_{L^2} \\ =& \langle (u(x)\frac{d}{dx}, v(x), c), [(fg' - f'g)\frac{d}{dx}, fb' - ga', \omega)\rangle_{L^2} \\ =& -\int_{S^1}(fg' - f'g)u(x)dx - \int_{S^1}(fb' - ga')v dx - c_1\int_{S^1} f'(x)g''(x)dx \\ &- c_2\int_{S^1}(f''(x)b(x) - g''(x)a(x))dx - 2c_3\int_{S^1} a(x)b'(x)dx\,.\end{aligned}$$

Since f, g, u are periodic functions, hence integrating by parts we obtain

$$\begin{aligned}R.H.S. =& \langle (2f'(x)u(x) + f(x)u'(x) + a'(x)v(x) - c_1 f'''(x) \\ &+ c_2 a''(x), f'(x)v(x) + f(x)v'(x) - c_2 f'' b(x) + 2c_3 a'(x), 0)\end{aligned}$$

□

The coadjoint action leaves the parameter space invariant. Let us consider a hyperplane $c_1 = -1, c_2 = c_3 = 0$.

Corollary 1.

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) + f''' \\ f'v(x) + f(x)v'(x) \\ 0 \end{pmatrix}$$

The Euler-Arnold equation is the Hamiltonian flow on the coadjoint orbit in $\hat{\mathcal{G}}^*$ [3], generated by the Hamiltonian

(12) $$H(\hat{u}) \equiv H(u, v) = \langle (u(x), v(x)), (u(x), v(x))\rangle\,,$$

given by

$$\frac{d\hat{u}}{dt} = -ad^*_{\hat{u}(t)}u(t)\,. \tag{13}$$

Let V be a vector space and assume that the Lie group G acts on the left by linear maps on V, thus G acts on the left on its dual space V^* [for details, see for example, 5].

Proposition 1. *Let $G \bigodot V$ be a semidirect product space (possibly infinite dimensional), equipped with a metric $\langle\cdot,\cdot\rangle$ which is right translation. A curve $t \to c(t)$ in $G \bigodot V$ is a geodesic of this metric if and only if $u(t) = d_{c(t)}R_{c(t)^{-1}}\dot{c}(t)$ satisfies the Euler-Arnold equation.*

Thus we prove the first part of our theorem.

3. H^1 METRIC AND INTEGRABLE EQUATION

Let us introduce H^1 norm on the algebra $\hat{\mathcal{G}}$

$$\begin{aligned}\langle \hat{f},\hat{g}\rangle_{H^1} \\ &= \int_{S^1} f(x)g(x)dx + \int_{S^1} a(x)b(x)dx \int_{S^1} \partial_x f(x)\partial_x g(x)dx \\ &\quad + \int_{S^1} \partial_x a(x)\partial_x b(x)dx + \alpha\cdot\beta\,,\end{aligned} \tag{14}$$

where $\hat{g}$ and $\hat{f}$ are as above.

Now we compute:

Lemma 2. *The coadjoint operator for H^1 norm is given by*

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} 2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'(1-\partial_x^2)v(x) - c_1f''' + c_2a'' \\ f'(1-\partial_x^2)v(x) + f(x)(1-\partial_x^2)v'(x) - c_2f'' + 2c_3a'(x) \\ 0 \end{pmatrix}$$

Proof. From the definition it follows that

$$\begin{aligned}\langle ad^*_{\hat{f}}\hat{u},\hat{g}\rangle_{H^1} \\ &= -\int_{S^1}(fg'-f'g)u(x)dx - \int_{S^1}(fb'-ga')vdx - c_1\int_{S^1} f'(x)g''(x)dx \\ &\quad - c_2\int_{S^1}(f''(x)b(x) - g''(x)a(x))dx - 2c_3\int_{S^1} a(x)b'(x)dx \\ &\quad - \int_{S^1}\partial_x(fg'-f'g)u(x)dx - \int_{S^1}\partial_x(fb'-ga')vdx\,.\end{aligned}$$

In the preceding section we have already computed the first five terms. After computing the last two terms by integrating by parts and using the fact that the functions $f(x), g(x), u(x)$ and $a(x), b(x), v(x)$ are periodic, this expression can be expressed as above.

Let us compute now the left hand side:

$$L.H.S. = \int_{S^1} (ad^*_{\hat{\xi}}\hat{q})\eta dx + \int_{S^1} (ad^*_{\hat{\xi}}\hat{q})'\eta' dx$$
$$= \int_{S^1} [(1-\partial^2) ad^*_{\hat{\xi}}\hat{q}]\eta dx\,.$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula. □

Corollary 2.

$$ad^*_{\hat{f}}\hat{u} =$$
$$\begin{pmatrix} 2f'(x)(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'(1-\partial_x^2)v(x) + f''' \\ f'(1-\partial_x^2)v(x) + f(x)(1-\partial_x^2)v'(x) \\ 0 \end{pmatrix}$$

Hence by applying the proposition 1 , we obtain the second part of our theorem.

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S.N. Bose National Centre for Basic Sciences
JD Block, Sector-3, Salt Lake
Calcutta-700091, INDIA