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CONJUGACY AND DISCONJUGACY CRITERIA FOR SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Conjugacy and disconjugacy criteria are established for the equation

$$u'' + p(t)u = 0 \,,$$

where $p :]-\infty, +\infty[\to]-\infty, +\infty[$ is a locally summable function.

Introduction

Consider the equation

$$u'' + p(t)u = 0 \,, \tag{1}$$

where $p : R \to R$ ($R =]-\infty, +\infty[$) is Lebesgue integrable on every compact subinterval of R. By a solution of the equation (1) we understand the function $u : R \to R$ which is absolutely continuous together with its first derivative on every segment contained in R, and almost everywhere satisfies the equation (1).

The equation (1) is said to be *conjugate* if it has a nontrivial solution with at least two zeros, and *disconjugate* otherwise. If there exists a nontrivial solution of (1) with a sequence of zeros tending to $+\infty$ (to $-\infty$), then the equation (1) is said to be *oscillatory* in the neighbourhood of $+\infty$ (in the neighbourhood of $-\infty$).

The history of the problem on conjugacy of (1) begins from the paper of S. W. Hawking and R. Penrose [8], where has been proved that conditions $p(t) \ge 0$ for $t \in R$ and $p \not\equiv 0$, guarantee conjugacy of (1). In [13] F. J. Tipler pointed out an interesting relevance of the study of conjugacy of (1) to general relativity and improved Hawking–Penrose's criterion, showing that (1) is conjugate whenever

$$\liminf_{t \to -\infty} \liminf_{\tau \to +\infty} \int_t^\tau p(s)\, ds > 0$$

is fulfilled. Later these results were generalized in various directions (see, e.g., [1, 2, 5, 6, 9 - 12]). However, the investigation of conjugacy of the equation (1) is not

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still completed. Below we establish new conditions for conjugacy and disconjugacy of the equation (1) which generalize and make the results obtained in [1, 5, 6, 8 - 10] more complete.

Put

$$c(t) = \frac{1}{|t|} \int_0^t \int_0^s p(\xi)\, d\xi\, ds \qquad \text{for } \ t \in R \setminus \{0\}\,.$$

According to Theorem 7.3 in [7, p. 367], if either

$$\lim_{t\to+\infty} c(t) = +\infty \quad \text{or} \quad -\infty < \liminf_{t\to+\infty} c(t) < \limsup_{t\to+\infty} c(t)\,,$$

resp. if either

$$\lim_{t\to-\infty} c(t) = +\infty \quad \text{or} \quad -\infty < \liminf_{t\to-\infty} c(t) < \limsup_{t\to-\infty} c(t)\,,$$

then the equation (1) is oscillatory in the neighbourhood of $+\infty$, resp. in the neighbourhood of $-\infty$. Therefore, below we suppose that there exist finite limits

$$c(-\infty) \overset{def}{=} \lim_{t\to-\infty} c(t)\,, \qquad c(+\infty) \overset{def}{=} \lim_{t\to+\infty} c(t)\,.$$

Denote

$$Q_1(t) = \int_0^t p(s)\, ds - c(+\infty)\,, \qquad Q_2(t) = \int_0^t p(s)\, ds + c(-\infty) \qquad \text{for } \ t \in R\,,$$

$$F_i(t) = \exp\left(2\int_0^t Q_i(s)\, ds\right) \qquad \text{for } \ t \in R\,, \quad i = 1, 2\,.$$

To formulate the main results, we need two propositions. The first one is the special case of Theorem 1.2 proved in [14], and the second one we prove in §3.

Proposition 1. *Let*

$$\int^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\, ds = +\infty\,, \quad \textit{resp.} \quad \int_{-\infty} \frac{Q_2^2(s)}{F_2(s)}\, ds = +\infty\,.$$

Then the equation (1) *is oscillatory in the neighbourhood of* $+\infty$, *resp. in the neighbourhood of* $-\infty$.

Proposition 2. *If*

$$\int_{-\infty} F_1(s)\, ds < +\infty\,, \quad \textit{resp.} \quad \int^{+\infty} F_2(s)\, ds < +\infty \tag{2}$$

and

$$\int_{-\infty} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds = +\infty\,, \quad \textit{resp.} \tag{3}$$
$$\int^{+\infty} \frac{\left[Q_2(s)\int_{s}^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds = +\infty\,,$$

then the equation (1) *is oscillatory in the neighbourhood of* $-\infty$*, resp. in the neighbourhood of* $+\infty$*.*

1. Main Results

Theorem 1. *Let* $p \not\equiv 0$ *and*

$$c(-\infty) + c(+\infty) \geq 0\,. \tag{4}$$

Then the equation (1) *is conjugate.*

Remark 1. It can be easily seen that Theorem 1 generalizes above-mentioned Tipler's result as well as Corollary 1 in [6] and Corollary 3.2 and Proposition 3 in [9].

Remark 2. From Theorem 1 and above-mentioned Theorem 7.3 in [7] it follows that at least one of the equations $u'' + p(t)u = 0$ and $u'' - p(t)u = 0$, where $p \not\equiv 0$, is conjugate whenever either

$$\liminf_{t\to-\infty} c(t) > -\infty \quad \text{or} \quad \limsup_{t\to-\infty} c(t) < +\infty$$

or

$$\liminf_{t\to+\infty} c(t) > -\infty \quad \text{or} \quad \limsup_{t\to+\infty} c(t) < +\infty\,.$$

According to Theorem 1, it is natural to assume in the following that

$$c(-\infty) + c(+\infty) < 0\,. \tag{5}$$

Note that the inequality (5) guarantees the conditions (2). Consequently, in view of Propositions 1 and 2 it is also natural to assume that

$$\int^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds < +\infty, \qquad \int_{-\infty} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds < +\infty\,, \tag{6}$$

$$\int_{-\infty} \frac{Q_2^2(s)}{F_2(s)}\,ds < +\infty, \qquad \int^{+\infty} \frac{\left[Q_2(s)\int_{s}^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds < +\infty\,. \tag{7}$$

Theorem 2. *Let the conditions* (5), (6), *and*

$$\sup\left\{\int_{-\infty}^{t} F_1(s)\,ds \int_{t}^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds + \frac{1}{\int_{-\infty}^{t} F_1(s)\,ds} \int_{-\infty}^{t} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds \;:\; t\in R\right\} > 1 \tag{8}$$

be fulfilled. Then the equation (1) *is conjugate.*

Theorem 2′. *Let the conditions* (5), (7), *and*

$$\sup\left\{\int_{t}^{+\infty} F_2(s)\,ds \int_{-\infty}^{t} \frac{Q_2^2(s)}{F_2(s)}\,ds + \frac{1}{\int_{t}^{+\infty} F_2(s)\,ds} \int_{t}^{+\infty} \frac{\left[Q_2(s)\int_{s}^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds \;:\; t\in R\right\} > 1$$

be fulfilled. Then the equation (1) *is conjugate.*

Finally we formulate disconjugacy theorems.

Theorem 3. *Let the conditions* (5) *and* (6) *hold. Let, moreover, at least one of the following three inequalities be fulfilled:*

$$\sup\left\{\int_{-\infty}^{t} F_1(s)\,ds \int_{t}^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,; \tag{9}$$

$$\sup\left\{\frac{1}{\int_{-\infty}^{t} F_1(s)\,ds} \int_{-\infty}^{t} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,; \tag{10}$$

$$\sup\left\{\int_{-\infty}^{t} F_1(s)\,ds \int_{t}^{+\infty} \frac{Q_1^2(s)}{F_1(s)}\,ds + \frac{1}{\int_{-\infty}^{t} F_1(s)\,ds} \int_{-\infty}^{t} \frac{\left[Q_1(s)\int_{-\infty}^{s} F_1(\xi)\,d\xi\right]^2}{F_1(s)}\,ds \;:\; t\in R\right\} \le \sqrt{2}-1\,. \tag{11}$$

Then the equation (1) *is disconjugate.*

Theorem 3′. *Let the conditions* (5) *and* (7) *hold. Let, moreover, at least one of the following three inequalities be fulfilled:*

$$\sup\left\{\int\limits_t^{+\infty} F_2(s)\,ds \int\limits_{-\infty}^{t} \frac{Q_2^2(s)}{F_2(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,;$$

$$\sup\left\{\frac{1}{\int_t^{+\infty} F_2(s)\,ds}\int\limits_t^{+\infty} \frac{\left[Q_2(s)\int_s^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds \;:\; t\in R\right\} \le \frac{1}{4}\,;$$

$$\sup\left\{\int\limits_t^{+\infty} F_2(s)\,ds \int\limits_{-\infty}^{t} \frac{Q_2^2(s)}{F_2(s)}\,ds + \right.$$
$$\left. +\frac{1}{\int_t^{+\infty} F_2(s)\,ds}\int\limits_t^{+\infty} \frac{\left[Q_2(s)\int_s^{+\infty} F_2(\xi)\,d\xi\right]^2}{F_2(s)}\,ds \;:\; t\in R\right\} \le \sqrt{2}-1\,.$$

Then the equation (1) *is disconjugate.*

2. Auxiliary Propositions

Lemma 1. *Let the equation* (1) *be disconjugate. Then it has a solution not having a zero.*

Proof. Let $\widetilde{v}_n$ and $\widetilde{w}_n$, where $n\in N$, be solutions of the equation (1) satisfying the initial conditions

$$\widetilde{v}_n(-n)=0,\qquad \widetilde{v}_n'(-n)=1\,,$$
$$\widetilde{w}_n(n)=0,\qquad \widetilde{w}_n'(n)=-1\,.$$

Evidently, $\widetilde{v}_n(0)\neq 0$ and $\widetilde{w}_n(0)\neq 0$ for $n\in N$. Put

$$v_n(t)=\frac{\widetilde{v}_n(t)}{\widetilde{v}_n(0)}\,,\qquad w_n(t)=\frac{\widetilde{w}_n(t)}{\widetilde{w}_n(0)}\qquad \text{for}\ \ t\in R,\quad n\in N\,.$$

It is clear that for any $n\in N$, v_n and w_n are solutions of the equation (1) and

$$\begin{aligned} &v_n(t)\operatorname{sgn}(t+n)>0 \qquad \text{for}\ \ t\in R\setminus\{-n\}\,,\\ &w_n(t)\operatorname{sgn}(t-n)<0 \qquad \text{for}\ \ t\in R\setminus\{n\}\,,\end{aligned}\tag{12}$$

$$v_n(0)=1,\qquad w_n(0)=1\,.\tag{13}$$

Since the equation (1) is disconjugate, we can easily verify that for $t\in R\setminus\{0\}$ and $n,m\in N$,

$$(v_{n+1}(t)-v_n(t))\operatorname{sgn} t<0\,,\qquad (w_{n+1}(t)-w_n(t))\operatorname{sgn} t>0\,,\tag{14}$$

$$(v_n(t)-w_m(t))\operatorname{sgn} t>0\,.\tag{15}$$

(14) and (15) result in

$$v_{n+1}'(0)<v_n'(0)\,,\qquad w_{n+1}'(0)>w_n'(0)\qquad \text{for}\ \ n\in N\,,\tag{16}$$

$$v_n'(0) > w_m'(0) \qquad \text{for } \ n, m \in N\,. \tag{17}$$

Put $\lambda = |w_1'(0)| + |v_1'(0)|$ and

$$M_a = \max\{w_1(t) : -a \le t \le 0\} + \max\{v_1(t) : 0 \le t \le a\}\,,$$

where $a > 0$. Then by (14), (16), and (17) we have

$$|v_n'(0)| \le \lambda, \qquad |w_n'(0)| \le \lambda \qquad \text{for } \ n \in N\,, \tag{18}$$

$$|v_n(t)| \le M_a, \qquad |w_n(t)| \le M_a \qquad \text{for } \ -a \le t \le a, \quad n > a. \tag{19}$$

Since v_n and w_n are solutions of the equation (1), in view of (18) and (19), we obtain

$$|v_n'(t)| \le \lambda_a\,, \qquad |w_n'(t)| \le \lambda_a \qquad \text{for } \ -a \le t \le a\,, \quad n > a\,, \tag{20}$$

where $\lambda_a = \lambda + M_a \int_{-a}^{a} |p(s)|\,ds$.

By (19) and (20), it is clear that the sequences $(v_k^{(i)})_{k=1}^{+\infty}$ and $(w_k^{(i)})_{k=1}^{+\infty}$, $i = 0, 1$, are uniformly bounded and equicontinuous in R (i.e., on every segment contained in R). Therefore, according to the Arzelà–Ascoli lemma, without loss of generality we can assume that

$$\lim_{k\to+\infty} v_k^{(i)}(t) = v^{(i)}(t)\,, \qquad \lim_{k\to+\infty} w_k^{(i)}(t) = w^{(i)}(t) \qquad , i = 0, 1$$

uniformly in R. It is easy to see that the functions v and w are solutions of the equation (1).

In view of (12), (14), and (15) we have

$$v(t) > 0 \qquad \text{for } \ t < 0, \qquad w(t) > 0 \qquad \text{for } \ t > 0\,,$$
$$(v(t) - w(t))\operatorname{sgn} t \ge 0 \qquad \text{for } \ t \ne 0\,.$$

Consequently, $v(t) > 0$ for $t \ne 0$. However, according to (13), $v(0) = 1$. Thus v has no zero. □

Lemma 2. *Let a solution u of the equation* (1) *have no zero in the interval $[a, +\infty[$, resp. in the interval $]-\infty, a]$. Then*

$$\int_a^{+\infty} \rho^2(s)\,ds < +\infty\,, \quad \text{resp.} \quad \int_{-\infty}^{a} \rho^2(s)\,ds < +\infty\,,$$

and the equality

$$\rho(t) = c(+\infty) - \int_0^t p(s)\,ds + \int_t^{+\infty} \rho^2(s)\,ds \qquad \text{for } \ t \ge a\,, \quad \text{resp.}$$

$$\rho(t) = -c(-\infty) + \int_t^0 p(s)\,ds - \int_{-\infty}^{t} \rho^2(s)\,ds \qquad \text{for } \ t \le a$$

holds, where $\rho(t) = \frac{u'(t)}{u(t)}$ for $t \ge a$, resp. for $t \le a$.

For the proof of this lemma see [7], p. 365, Lemma 7.1.

Lemma 3. *Let the function* $g\ :]a,b[\to [0,+\infty[$ *be integrable on every segment contained in* $]a,b]$, *and*

$$\int_a^b (s-a)^2 g(s)\,ds = +\infty\,.$$

Then every nontrivial solution of the equation $v'' + g(x)v = 0$ *has a sequence of zeros tending to* a.

Lemma 3 immediately follows from Theorem 1.1 in [3].

3. Proofs

Proof of Proposition 2. Put

$$\varphi(t) = \int_0^t F_1(s)\,ds \qquad \text{for } \ t \in R\,. \tag{21}$$

According to (2) there exists a finite limit

$$a \overset{def}{=} \lim_{t\to-\infty} \varphi(t)\,, \tag{22}$$

and $a < 0$.

Define the function g by the equalities

$$g(x) = \frac{Q_1^2(t)}{F_1^2(t)}, \qquad x = \varphi(t) \qquad \text{for } \ t \in R\,, \tag{23}$$

and in the interval $]a, 0[$ consider the equation

$$v'' + g(x)v = 0\,. \tag{24}$$

In view of (23), from (3) we get

$$\int_a^0 (x-a)^2 g(x)dx = +\infty\,.$$

Consequently, by Lemma 3, every nontrivial solution of the equation (24) has a sequence of zeros tending to a. It can be dirrectly verify that if v is a solution of the equation (24), then the function u defined by the equality

$$u(t) = v(\varphi^{-1}(t))\exp\left(-\int_0^t Q_1(s)\,ds\right) \qquad \text{for } \ t \in R \tag{25}$$

is a solution of the equation (1). Obviously, u has a sequence of zeros tending to $-\infty$. □

Proof of Theorem 1. According to Lemma 1, to prove the theorem it is sufficient to show that the equation (1) has no solution without zeros. Assume the contrary.

Suppose u is a solution of the equation (1) with no zero in R. Then, by virtue of Lemma 2, the equalities

$$\rho(t) = c(+\infty) - \int_0^t p(s)\,ds + \int_t^{+\infty} \rho^2(s)\,ds \qquad \text{for } t \geq 0\,,$$

$$\rho(t) = -c(-\infty) + \int_t^0 p(s)\,ds - \int_{-\infty}^t \rho^2(s)\,ds \qquad \text{for } t \leq 0$$

are fulfilled, where $\rho(t) = \frac{u'(t)}{u(t)}$ for $t \in R$. Set in the latter equalities $t = 0$. If we subtract the second equality from the first one, we obtain

$$c(+\infty) + c(-\infty) = -\int_{-\infty}^{+\infty} \rho^2(s)\,ds\,. \tag{26}$$

Since $p \not\equiv 0$, then $\rho \not\equiv 0$ and, consequently, $\int_{-\infty}^{+\infty} \rho^2(s)\,ds > 0$. Hence, on account of (4) and (26), we get the contradiction $0 < 0$. □

Proof of Theorem 2. According to Theorem 1.1 in [4], if

$$\limsup_{t\to+\infty} \frac{t}{\ln t}\Big(c(+\infty) - c(t)\Big) > \frac{1}{4}\,,$$

then the equation (1) is oscillatory in the neighbourhood of $+\infty$. Therefore we assume that

$$\limsup_{t\to+\infty} \frac{t}{\ln t}\Big(c(+\infty) - c(t)\Big) \leq \frac{1}{4}\,. \tag{27}$$

Define the function φ by (21). Due to (27), it is easy to see that

$$\lim_{t\to+\infty} \varphi(t) = +\infty\,. \tag{28}$$

According to (5) we have

$$\lim_{t\to-\infty} \frac{|t|}{\ln|t|}\Big(-c(+\infty) - c(t)\Big) = +\infty\,. \tag{29}$$

Therefore there exists a finite limit (22). Consequently, the function φ monotonicly transforms R into $]a, +\infty[$.

Define the function g by (23), and in the interval $]a, +\infty[$ consider the equation (24). It is clear that if v is a solution of the equation (24), then the function u defined by (25) is a solution of the equation (1). Therefore, according to Lemma 1, to prove the theorem it is sufficient to show that the equation (24) has no solution without zeros in $]a, +\infty[$. Assume the contrary, let v be a solution of the equation (24) with no zero in $]a, +\infty[$. Put $\sigma(x) = \frac{v'(x)}{v(x)}$ for $x > a$. Taking into account (6) and Lemma 2, we find

$$\sigma(x)=\int\limits_x^{+\infty} g(\tau)d\tau+\int\limits_x^{+\infty}\sigma^2(\tau)d\tau \qquad \text{for } x>a\,. \tag{30}$$

Evidently, $\sigma'(x)=-g(x)-\sigma^2(x)$ for $x>a$. Multiplying both sides of the last equality by $x-\lambda$, where $\lambda>a$, and integrating from λ to x, we get

$$\begin{aligned}\sigma(x)&=-\frac{1}{(x-\lambda)^2}\int\limits_\lambda^x(\tau-\lambda)^2g(\tau)d\tau+\frac{1}{(x-\lambda)^2}\int\limits_\lambda^x(\tau-\lambda)\sigma(\tau)(2-(\tau-\lambda)\sigma(\tau))d\tau\\&\le-\frac{1}{(x-\lambda)^2}\int\limits_\lambda^x(\tau-\lambda)^2g(\tau)d\tau+\frac{1}{x-\lambda}\qquad\text{for } x>\lambda\,.\end{aligned}$$

Hence, on account of (6) and (23), it immediately follows

$$\sigma(x)\le-\frac{1}{(x-a)^2}\int\limits_a^x(\tau-a)^2g(\tau)d\tau+\frac{1}{x-a}\qquad\text{for } x>a\,. \tag{31}$$

From (30) and (31) we have

$$(x-a)\int\limits_x^{+\infty}g(\tau)d\tau+\frac{1}{(x-a)^2}\int\limits_a^x(\tau-a)^2g(\tau)d\tau\le1-(x-a)\int\limits_x^{+\infty}\sigma^2(\tau)d\tau\quad\text{for } x>a\,.$$

Consequently,

$$\sup\left\{(x-a)\int\limits_x^{+\infty}g(\tau)d\tau+\frac{1}{(x-a)^2}\int\limits_a^x(\tau-a)^2g(\tau)d\tau : x>a\right\}\le1\,.$$

Taking now into account (8) and (23), we obtain the contradiction $1<1$. □

The proof of Theorem 2′ is analogous to that of Theorem 2.

Proof of Theorem 3. Define the function φ by (21) and show that (28) is fulfilled. Indeed, if there exists $M>0$ such that

$$\varphi(t)\le M\qquad\text{for } t>0\,, \tag{32}$$

then according to (6) and the Hölder inequality we have

$$\left|\int\limits_0^t Q_1(s)\,ds\right|\le\sqrt{\varphi(t)}\sqrt{\int\limits_0^t\frac{Q_1^2(s)}{F_1(s)}\,ds}\le\sqrt{M}\sqrt{\int\limits_0^{+\infty}\frac{Q_1^2(s)}{F_1(s)}\,ds}<+\infty\qquad\text{for } t>0\,,$$

which contradicts (32). Consequently, (28) is fulfilled.

In view of (5) we have (29). Therefore there exists a finite limit (22). Thus the function φ monotonicly transforms R into $]a,+\infty[$.

Define the function g by (23), and in the interval $]a,+\infty[$ consider the equation (24). It is clear that if v is a solution of the equation (24), then the function u

defined by (25) is a solution of the equation (1). Therefore, to prove the theorem it is sufficient to show that the equation (24) has a solution with no zero in $]a,+\infty[$. However, to prove this fact it is sufficient to show that there exists a function $\rho :]a,+\infty[\to R$ which is absolutely continuous on every segment contained in $]a,+\infty[$ and satisfies the inequality

$$\rho'(x) \le -g(x) - \rho^2(x) \qquad \text{for } x > a\,. \tag{33}$$

Suppose that (9) is fulfilled. On account of (23), from (9) we obtain

$$(x-a)\int_x^{+\infty} g(s)\,ds \le \frac{1}{4} \qquad \text{for } x > a\,.$$

It can be easily verified that the function

$$\rho(x) = \frac{1}{4(x-a)} + \int_x^{+\infty} g(s)\,ds \qquad \text{for } x > a$$

satisfies the inequality (33).

Now suppose that (10) is fulfilled. Then in view of (23) we have

$$\frac{1}{x-a}\int_a^x (s-a)^2 g(s)\,ds \le \frac{1}{4} \qquad \text{for } x > a.$$

It is easy to verify that the function

$$\rho(x) = \frac{3}{4(x-a)} - \frac{1}{(x-a)^2}\int_a^x (s-a)^2 g(s)\,ds \qquad \text{for } x > a$$

satisfies the inequality (33).

Finally suppose that (11) is fulfilled. Then by (23) we get

$$(x-a)\int_x^{+\infty} g(s)\,ds + \frac{1}{x-a}\int_a^x (s-a)^2 g(s)\,ds \le \sqrt{2}-1 \qquad \text{for } x > a\,.$$

We can easily verify that the function

$$\rho(x) = \frac{1}{2(x-a)} + \frac{1}{2}\left(\int_x^{+\infty} g(s)\,ds - \frac{1}{(x-a)^2}\int_a^x (s-a)^2 g(s)\,ds\right) \qquad \text{for } x > a$$

satisfies the inequality (33). □

Theorem 3′ can be proved analogously.

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