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THREE-DIMENSIONAL CURVATURE HOMOGENEOUS HYPERSURFACES

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Abstract. This paper is motivated by the open problem whether a three-dimensional curvature homogeneous hypersurface of a real space form is locally homogeneous or not. We give some partial positive answers.

1. Introduction

Let (M, g) be a n-dimensional Riemannian manifold with curvature tensor R. (M, g) is said to be locally homogeneous if for each $p, q \in M$ there exists a local isometry of a neighbourhood of p onto a neighbourhood of q which maps p to q. Clearly such a space has the following property:

For each $p, q \in M$ there exists a linear isometry of the tangent space T_pM on the tangent space T_qM which maps R_p in R_q, that is, it preserves the curvature tensor.

According to I. M. Singer [13], a Riemannian manifold having this last property is called *curvature homogeneous.*

It is well known that the class of locally homogeneous spaces is strictly contained in the class of curvature homogeneous spaces. K. Sekigawa gave in [12] an infinite family of isometry classes of irreducible complete Riemannian metrics on $\mathbb{R}^3$ which are curvature homogeneous but not locally homogeneous (see also [14]). Since then, many other authors gave a lot of examples of curvature homogeneous spaces which are not locally homogeneous (see [1] for more complete references). For this reason, it is interesting to investigate the following problem:

Under which conditions does curvature homogeneity imply local homogeneity?

In this paper we give a contribution to this question in the class of hypersurfaces of a space of constant sectional curvature (i.e., a *space form*).

We recall that an *isoparametric hypersurface* in a space form is a hypersurface having constant principal curvatures (E. Cartan). Such hypersurfaces are always

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curvature homogeneous as it follows easily from the Gauss equation. H. Ozeki and M. Takeuchi ([10], [11]) found examples of isoparametric hypersurfaces of the unit sphere which are not homogeneous and successively much more examples were constructed by D. Ferus, H. Karcher, H.-F. Münzner [8]. K. Tsukada [16] stated the following problem: *determine all curvature homogeneous hypersurfaces* M^n *immersed in a standard space form* $M^{n+1}(\tilde{c})$, where $M^{n+1}(\tilde{c})$ is the Euclidean sphere $S^{n+1}(\tilde{c})$ if $\tilde{c} > 0$, the Euclidean space E^{n+1} if $\tilde{c} = 0$ and the hyperbolic space $H^{n+1}(\tilde{c})$ if $\tilde{c} < 0$. In the same paper he proved:

(i) A curvature homogeneous hypersurface M^n of the Euclidean space E^{n+1} is locally symmetric (and hence, locally homogeneous) for all $n \geq 2$.

(ii) A curvature homogeneous hypersurface M^n of the unit sphere $S^{n+1}(\tilde{c})$, $n \geq 4$, is either a space of constant curvature or it is isoparametric.

(iii) A curvature homogeneous hypersurface M^n of the hyperbolic space $H^{n+1}(\tilde{c})$, $n \geq 5$, is locally homogeneous. For $n = 4$ there is, up to local congruence, exactly one proper curvature homogeneous hypersurface in $H^5(\tilde{c})$.

Quite surprizingly, for $n = 3$ the problem whether curvature homogeneous hypersurfaces of $S^4(\tilde{c})$ or $H^4(\tilde{c})$ are locally homogeneous or not, is still open! (see [1], p. 255).

Motivated by this unsolved question, in this paper we consider three-dimensional curvature homogeneous hypersurfaces in a space form $M^4(\tilde{c})$ with $\tilde{c} \neq 0$. We first consider isoparametric hypersurfaces. We prove that *a three-dimensional isoparametric hypersurface of a real space form is locally homogeneous* (we remark that this can also be obtained combining some results of [4] and [5]). Using this fact, we prove:

A curvature homogeneous hypersurface M^3 *of a space form* $M^4(\tilde{c})$ *(*$\tilde{c} \neq 0$*) is locally homogeneous or it has the following properties:*

(a) *just two of the Ricci eigenvalues are equal;*

(b) *the shape operator* S *has rank* 2*;*

(c) M *is not isoparametric.*

Properties (a) and (b) of this result specify under which conditions we are not able to decide about local homogeneity of M^3. Under some additional assumptions we can conclude that M^3 is locally homogeneous. In particular, M^3*is locally homogeneous if one of the following conditions holds:*

1) M^3 *satisfies* (a) *and* (b) *and the mean curvature* H *is constant along the geodesic foliation generated by a unit vector field belonging to the kernel of* S*;* (more specifically, in this case M^3 is a *Cartan's minimal hypersurface* [15])

2) M^3 *is compact and* $\tilde{c} < 0$*;*

3) M^3 *is compact,* $\tilde{c} > 0$ *and the scalar curvature of* M^3 *satisfies* $\tau \geq 2\tilde{c}$*;*

4) $\tilde{c} > 0$ *and* $\tau \geq 6\tilde{c}$*;*

5) M^3 *is ball-homogeneous.*

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2. Preliminaries on hypersurfaces of a space form

Let (M, g) be a n-dimensional connected Riemannian hypersurface of a $(n+1)$-dimensional real space form $M^{n+1}(\tilde{c})$. We denote by ∇ the Levi-Civita connection and by R the curvature tensor of M, taken with the sign convention

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] .$$

Moreover, ϱ and τ denote the Ricci tensor and the scalar curvature of M, respectively. Let h be the second fundamental form of M $(h(X,Y) = g(SX,Y)$, where S is the shape operator of $M)$. Let U be the open subset of M where $S \neq 0$ and V the open subset of points $p \in M$ such that $S = 0$ in a neighbourhood of p. Then, $U \cup V$ is a dense open subset of M. For any $p \in U \cup V$ there exists a local orthonormal basis of smooth fields of eigenvectors of S in a neighbourhood of p. On U we put $Se_i = \lambda_i e_i$ for all i. With respect to $\{e_1, ..., e_n\}$, the components of R are given by the *Gauss equation*

$$R_{ijkh} = \tilde{c}(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}) + h_{ik}h_{jh} - h_{ih}h_{jk} = (\tilde{c} + \lambda_i\lambda_j)(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}) .$$

The components of the Ricci tensor are given by

$$\varrho_{ij} = \{(n-1)\tilde{c} + H\lambda_i - \lambda_i\lambda_j\}\delta_{ij} , \tag{1}$$

$H = \sum_i \lambda_i$ being the *mean curvature* of M. M is said to be *minimal* if $H = 0$ and *totally geodesic* if $h = 0$.

Note that (1) implies that $\{e_1, ..., e_n\}$ is also a local orthonormal moving frame of eigenvectors for the Ricci operator.

The components of ∇h with respect to $\{e_1, ..., e_n\}$ are given by

$$\begin{cases} \nabla_i h_{jj} = e_i(\lambda_j) , \\ \nabla_i h_{jk} = (\lambda_j - \lambda_k) g(\nabla_{e_i} e_j, e_k) \quad if\ j \neq k , \end{cases} \tag{2}$$

and satisfy the *Codazzi equation*

$$\nabla_i h_{jk} = \nabla_j h_{ik} \quad \text{for} \quad \text{all } i, j, k = 1, .., n . \tag{3}$$

The components of $\nabla \varrho$ are given by

$$\begin{cases} \nabla_i \varrho_{jj} = e_i(\varrho_j) , \\ \nabla_i \varrho_{jk} = (\varrho_j - \varrho_k) g(\nabla_{e_i} e_j, e_k) \quad if\ j \neq k . \end{cases} \tag{4}$$

3. Homogeneity of a three-dimensional curvature homogeneous hypersurface

Let M be a three-dimensional Riemannian manifold immersed in a space form $M^4(\tilde{c})$ with $\tilde{c} \neq 0$ and $\{e_1, e_2, e_3\}$ a local orthonormal moving frame of eigenvectors of the shape operator S of M. From (1) and for $n = 3$, we get that the Ricci eigenvalues of M are given by

$$\left\{ \begin{array}{l} \varrho_1 = 2\tilde{c} + \lambda_1\lambda_2 + \lambda_1\lambda_3\,, \\ \varrho_2 = 2\tilde{c} + \lambda_1\lambda_2 + \lambda_2\lambda_3\,, \\ \varrho_3 = 2\tilde{c} + \lambda_1\lambda_3 + \lambda_2\lambda_3\,. \end{array} \right. \tag{5}$$

Proposition 1. *A three-dimensional isoparametric hypersurface M of a space form $M^4(\tilde{c})$ is locally homogeneous.*

Proof. From (5) it follows that an isoparametric hypersurface is curvature homogeneous. Then, if $\tilde{c} = 0$ the result is known since in this case a curvature homogeneous hypersurface is locally symmetric (see [16]). So, in what follows we suppose $\tilde{c} \neq 0$. Since the principal curvatures of M are constant, we consider the following cases: (I) $\lambda_1\lambda_2\lambda_3 \neq 0$; (II) $\lambda_1\lambda_2 \neq 0$, $\lambda_3 = 0$; (III) $\lambda_2 = \lambda_3 = 0$. □

Case I

If $\varrho_1 = \varrho_2 = \varrho_3$, then M is an Einstein manifold and so, it has constant sectional curvature. In particular, M is locally homogeneous.

If $\varrho_1 = \varrho_2 \neq \varrho_3$, we prove that $\nabla\varrho = 0$, that is, M is locally symmetric. Using (4), we get that the only possible non-vanishing components of $\nabla\varrho$ are given by

$$\nabla_k\varrho_{i3} = (\varrho_i - \varrho_3)g(\nabla_{e_k}e_i, e_3), \quad \text{for } k = 1,2,3 \text{ and } i = 1,2.$$

On the other hand, $\varrho_1 = \varrho_2 \neq \varrho_3$, $\lambda_1\lambda_2\lambda_3 \neq 0$ and (5) imply $\lambda_1 = \lambda_2 \neq \lambda_3$. Moreover, using (2) and the Codazzi equation (3), we have

$$\left\{ \begin{array}{l} (\lambda_i - \lambda_3)g(\nabla_{e_i}e_i, e_3) = \nabla_i h_{i3} = \nabla_3 h_{ii} = 0, \\ (\lambda_i - \lambda_3)g(\nabla_{e_3}e_i, e_3) = \nabla_3 h_{i3} = \nabla_i h_{33} = 0, \quad \text{for } i = 1,2. \end{array} \right.$$

Therefore, since $\lambda_i \neq \lambda_3$, $g(\nabla_{e_i}e_i, e_3) = g(\nabla_{e_3}e_i, e_3) = 0$ and so, applying (4), we get $\nabla_i\varrho_{i3} = \nabla_3\varrho_{i3} = 0$, for $i = 1,2$. Hence, it remains to show that $\nabla_1\varrho_{23} = \nabla_2\varrho_{13} = 0$. Now, since $\lambda_1 = \lambda_2$ and $\lambda_2 \neq \lambda_3$, we have

$$(\lambda_2 - \lambda_3)g(\nabla_{e_1}e_2, e_3) = \nabla_1 h_{23} = \nabla_3 h_{12} = (\lambda_1 - \lambda_2)g(\nabla_{e_3}e_1, e_2) = 0$$

and hence, $g(\nabla_{e_1}e_2, e_3) = 0$. In the same way, $\nabla_2 h_{13} = \nabla_3 h_{12}$ implies $g(\nabla_{e_2}e_1, e_3) = 0$. Therefore, $\nabla_1\varrho_{23} = \nabla_2\varrho_{13} = 0$. Thus, we can conclude that $\nabla\varrho = 0$ and, since $\dim M = 3$, M is locally symmetric. In particular, M is locally homogeneous.

If $\varrho_1 \neq \varrho_2 \neq \varrho_3 \neq \varrho_1$, we show that the 1-form $\varrho\cdot\nabla\varrho$ vanishes. Then Theorems A and 3.2 of [18] imply that M is locally homogeneous.

By definition, $(\varrho\cdot\nabla\varrho)_i = \sum_{a,b}\varrho_{ab}\nabla_b\varrho_{ia}$. In our case, since $\{e_1, e_2, e_3\}$ is a basis of eigenvectors for ϱ, we have $(\varrho\cdot\nabla\varrho)_i = \sum_a \varrho_a\nabla_a\varrho_{ia}$. From (4) we get $\nabla_a\varrho_{ia} = 0$ if $a = i$ and $\nabla_a\varrho_{ia} = (\varrho_i - \varrho_a)g(\nabla_{e_a}e_i, e_a)$ for $a \neq i$. In this last case, the Codazzi equation (3) gives

$$(\lambda_i - \lambda_a)g(\nabla_{e_a}e_i, e_a) = \nabla_a h_{ia} = \nabla_a h_{ii} = 0\,.$$

Since all ϱ_i are distinct, all λ_i are distinct. Then, $\varrho\cdot\nabla\varrho = 0$ and hence, M is locally homogeneous.

Case II

We first prove that we can not have $\lambda_1 = \lambda_2$. In fact, we can apply Cartan's formula for the principal curvatures of M (see [7]):

$$\sum_{j\neq i, j=1}^{g} m_j \frac{\tilde{c}+\lambda_i\lambda_j}{\lambda_j-\lambda_i} = 0 \quad \text{for all i}, \tag{6}$$

where $\lambda_1, .., \lambda_g$ are the distinct principal curvatures and m_j the multiplicity of λ_j. In our case, from (6) it follows, for $i = 1$,

$$\frac{\tilde{c}+\lambda_1\lambda_3}{\lambda_3-\lambda_1} = 0$$

and from $\lambda_3 = 0$ we get $\tilde{c} = 0$, which contradicts our assumption.

So, $\lambda_1 \neq \lambda_2$. We shall prove that $||\nabla\varrho||^2$ is constant on M. Then, since $\varrho_1 = \varrho_2 \neq \varrho_3$, Proposition 7.3 of [9] (which holds in general for $\varrho_3 \neq 0$) implies that M is locally homogeneous.

From (4), taking into account that $\varrho_1 = \varrho_2$, we get that the only possible non-vanishing components of $\nabla\varrho$ are $\nabla_k\varrho_{i3} = (\varrho_i - \varrho_3)g(\nabla_{e_k}e_i, e_3)$, for $k = 1, 2, 3$ and $i = 1, 2$. Since $\lambda_3 = 0$ and $\lambda_1\lambda_2 \neq 0$, using the Codazzi equation (3), we have

$$\begin{cases} \lambda_i g(\nabla_{e_i}e_i, e_3) = \nabla_i h_{i3} = \nabla_3 h_{ii} = 0, \\ \lambda_i g(\nabla_{e_3}e_i, e_3) = \nabla_3 h_{i3} = \nabla_i h_{33} = 0, \end{cases}$$

that is, $g(\nabla_{e_i}e_i, e_3) = g(\nabla_{e_3}e_i, e_3) = 0$ and so, by (4), $\nabla_i\varrho_{i3} = \nabla_3\varrho_{i3} = 0$ for $i = 1, 2$. So, we only have to consider $\nabla_2\varrho_{13} = (\varrho_1 - \varrho_3)\beta$ and $\nabla_1\varrho_{23} = (\varrho_2 - \varrho_3)\gamma$, where $\beta = g(\nabla_{e_2}e_1, e_3)$ and $\gamma = g(\nabla_{e_1}e_2, e_3)$. Since $\lambda_3 = 0$, formulas (2) and (3) also give

$$\lambda_1 g(\nabla_{e_2}e_1, e_3) = \nabla_2 h_{13} = \nabla_1 h_{23} = \lambda_2 g(\nabla_{e_1}e_2, e_3),$$

that is,

$$\lambda_1\beta = \lambda_2\gamma\,. \tag{7}$$

Next, we shall compute the components of R with respect to $\{e_1, e_2, e_3\}$. Using (2) to compute $\nabla_i h_{jk}$ and applying the Codazzi equation (3), we obtain that the only non-vanishing covariant derivatives are

$$\begin{array}{ll} \nabla_{e_1}e_2 = \gamma e_3\,, & \nabla_{e_1}e_3 = -\gamma e_2\,, \\ \nabla_{e_2}e_1 = \beta e_3\,, & \nabla_{e_2}e_3 = -\beta e_1\,, \\ \nabla_{e_3}e_1 = \dfrac{\lambda_1\beta}{\lambda_1-\lambda_2}e_2\,, & \nabla_{e_3}e_2 = -\dfrac{\lambda_1\beta}{\lambda_1-\lambda_2}e_1\,. \end{array}$$

So, we get

$$\begin{aligned} R_{131} &= \nabla_{[e_1,e_3]}e_1 - \nabla_{e_1}\nabla_{e_3}e_1 + \nabla_{e_3}\nabla_{e_1}e_1 = \\ &= -e_1\left(\frac{\lambda_2\gamma}{\lambda_1-\lambda_2}\right)e_2 - \frac{2\lambda_1\beta\gamma}{\lambda_1-\lambda_2}e_3 \end{aligned}$$

and hence,

$$R_{1313} = -\frac{2\lambda_1\beta\gamma}{\lambda_1 - \lambda_2}\,.$$

On the other hand, the Gauss equation gives $R_{1313} = \tilde{c}$. So, $\beta\gamma$ is constant and this fact, together with (7), implies that β and γ are constant on M. Therefore, since

$$||\nabla\varrho||^2 = 2(\varrho_1 - \varrho_3)^2(\beta^2 + \gamma^2)\,,$$

we can conclude that $||\nabla\varrho||^2$ is constant and hence, M is locally homogeneous.

Case III

From (5) it follows at once that $\varrho_1 = \varrho_2 = \varrho_3$. Then, M is Einsteinian and so, M has constant sectional curvature. In particular it is locally homogeneous. □

Remark. E. Cartan [4] proved that isoparametric hypersurfaces of $H^{n+1}(\tilde{c})$ and compact isoparametric hypersurfaces of $S^{n+1}(\tilde{c})$ are locally homogeneous (see also [17]). On the other hand, in the spherical case, any isoparametric hypersurface is an open part of a compact hypersurface [5, p.239]. So, the result of Proposition 1 can also follow from this fact.

Theorem 2. *A curvature homogeneous hypersurface M of a space form $M^4(\tilde{c})$ ($\tilde{c} \neq 0$) is either locally homogeneous or it has the following properties:*

(a) *just two of the Ricci eigenvalues are equal;*

(b) *the shape operator S has rank 2;*

(c) *M is not isoparametric.*

Proof. Since M is curvature homogeneous, it has constant Ricci eigenvalues. Therefore, the possible cases are the following:

(A) $\varrho_1 = \varrho_2 = \varrho_3$;

(B) $\varrho_1 \neq \varrho_2 \neq \varrho_3 \neq \varrho_1$;

(C) $\varrho_1 = \varrho_2 \neq \varrho_3$.

We shall treat these cases separately.

Case A. M is Einsteinian and hence it has constant sectional curvature. In particular, M is locally homogeneous.

Case B. Since ϱ_1, ϱ_2 and ϱ_3 are constant and all distinct, from (5) it follows easily that λ_1, λ_2 and λ_3 are all distinct and $\lambda_i \neq 0$ for all i. Moreover, (5) also gives

$$(8)\qquad \begin{cases} \lambda_1\lambda_2 = \frac{1}{2}(\varrho_1 + \varrho_2 - \varrho_3 - \tilde{c})\,, \\ \lambda_1\lambda_3 = \frac{1}{2}(\varrho_1 + \varrho_3 - \varrho_2 - \tilde{c})\,, \\ \lambda_2\lambda_3 = \frac{1}{2}(\varrho_2 + \varrho_3 - \varrho_1 - \tilde{c}) \end{cases}$$

and hence $(\lambda_1\lambda_2\lambda_3)^2$ is constant. Since $\lambda_i \neq 0$ for all i, the constancy of $\lambda_i\lambda_j$, $i \neq j$, implies that λ_1, λ_2 and λ_3 are constant on U and hence, on M. Therefore, M is isoparametric and Proposition 1 implies that M is locally homogeneous.

Case C. Since $\varrho_1 = \varrho_2$, from (5) we get

$$(\lambda_1 - \lambda_2)\lambda_3 = 0 \qquad \text{on} \quad M\,. \tag{9}$$

Note that either $\lambda_3 = 0$ or $\lambda_3 \neq 0$ everywhere on M. In fact, suppose $\lambda_3(p) = 0$ for some $p \in M$. Then, $\varrho_3 = \lambda_1\lambda_3 + \lambda_2\lambda_3 + 2\tilde{c} = \varrho_3(p) = 2\tilde{c}$ and hence, $(\lambda_1 + \lambda_2)\lambda_3 = 0$. Now if there exists $q \in M$ such that $\lambda_3(q) \neq 0$, then $(\lambda_1 + \lambda_2)(q) = 0$ and from (9) we also get $(\lambda_1 - \lambda_2)(q) = 0$ and hence, $\lambda_1(q) = \lambda_2(q) = 0$ and so, $\varrho_1(q) = \varrho_2(q) = \varrho_3(q) = 2\tilde{c}$, which is a contradiction.

Thus, we have to consider the following subcases:

(C1) $\lambda_3 \neq 0$.

From (9) it follows $\lambda_1 = \lambda_2$ on M and hence the constancy of ϱ_i by (5) implies that $\lambda_1 = \lambda_2$ and λ_3 are constant on M. So, M is isoparametric and Proposition 1 implies that M is locally homogeneous.

(C2) $\lambda_3 = 0$.

First, we note that $(\lambda_1\lambda_2)(p) \neq 0$ for all $p \in M$, otherwise $\varrho_1 = \varrho_2 = \varrho_3$. So, rank$S = 2$ and $\varrho_1 = \varrho_2 \neq \varrho_3$. If M is isoparametric, then it is locally homogeneous because of Proposition 1 and this ends the proof. □

In the case of a curvature homogeneous hypersurface of $M^4(\tilde{c})$ satisfying properties (a) and (b), let us consider the geodesic foliation generated by a local unit vector field e_3 belonging to the kernel of the shape operator.

Theorem 3. *Let M be a three-dimensional curvature homogeneous hypersurface of a space form $M^4(\tilde{c})$ ($\tilde{c} \neq 0$), with the properties* (a) *and* (b). *If some derivative $e_3^k(H)$ of the mean curvature H vanishes, where e_3 is a local unit vector field of the kernel of S, then M is minimal, $\tilde{c} > 0$ and the principal curvatures of M are $\lambda_1 = \sqrt{3\tilde{c}}$, $\lambda_2 = -\sqrt{3\tilde{c}}$ and $\lambda_3 = 0$. In particular, M is locally homogeneous.*

Proof. Since M satisfies the properties (a) and (b), we have to study the case

$$\lambda_3 = 0, \qquad \lambda_1\lambda_2 \neq 0 \quad \textit{everywhere}.$$

We first note that the case $\lambda_1 = \lambda_2$ can not occur. In fact, if $\lambda_1 = \lambda_2$ everywhere, the constancy of ϱ_i implies $\lambda_1 = \lambda_2 =$ constant and we noted in the proof of Proposition 1 that this is not possible.

So, we have to consider only the case

$$\lambda_3 = 0, \quad \lambda_1 \neq \lambda_2, \quad \lambda_1 \neq 0 \textit{ and } \lambda_2 \neq 0 \textit{ on a neighbourhood } W \subset U\,. \tag{10}$$

Since $\varrho_1 = \varrho_2$, from (4) we obtain that on W the only possible non-vanishing components of $\nabla\varrho$ are given by $\nabla_k\varrho_{i3} = (\varrho_i - \varrho_3)g(\nabla_{e_k}e_i, e_3)$, for $k = 1, 2, 3$ and $i = 1, 2$.

The constancy of the scalar curvature τ gives

$$0 = \frac{1}{2}\nabla_{e_3}\tau = \sum_r \nabla_r\varrho_{r3} = \nabla_1\varrho_{13} + \nabla_2\varrho_{23}$$

from which it follows that $g(\nabla_{e_1}e_3, e_1) = -g(\nabla_{e_2}e_3, e_2)$.

Next, using (2) to compute $\nabla_i h_{jk}$ and applying the Codazzi equation (3), we get

$$\nabla_1 h_{12} = \nabla_2 h_{11} \Rightarrow (\lambda_1 - \lambda_2) g(\nabla_{e_1} e_1, e_2) = -(\lambda_1 - \lambda_2) g(\nabla_{e_1} e_2, e_1) = e_2(\lambda_1)\,,$$
$$\nabla_1 h_{13} = \nabla_3 h_{11} \Rightarrow \lambda_1 g(\nabla_{e_1} e_1, e_3) = -\lambda_1 g(\nabla_{e_1} e_3, e_1) = e_3(\lambda_1)\,,$$
$$\nabla_2 h_{12} = \nabla_1 h_{22} \Rightarrow (\lambda_1 - \lambda_2) g(\nabla_{e_2} e_1, e_2) = -(\lambda_1 - \lambda_2) g(\nabla_{e_2} e_2, e_1) = e_1(\lambda_2)\,,$$
$$\nabla_2 h_{23} = \nabla_3 h_{22} \Rightarrow \lambda_2 g(\nabla_{e_2} e_2, e_3) = -\lambda_2 g(\nabla_{e_2} e_3, e_2) = e_3(\lambda_2)\,,$$
$$\nabla_3 h_{13} = \nabla_1 h_{33} \Rightarrow \lambda_1 g(\nabla_{e_3} e_1, e_3) = -\lambda_1 g(\nabla_{e_3} e_3, e_1) = 0\,,$$
$$\nabla_3 h_{23} = \nabla_2 h_{33} \Rightarrow \lambda_2 g(\nabla_{e_3} e_2, e_3) = -\lambda_2 g(\nabla_{e_3} e_3, e_2) = 0\,,$$
$$\nabla_1 h_{23} = \nabla_2 h_{13} = \nabla_3 h_{12}$$
$$\Rightarrow \lambda_2 g(\nabla_{e_1} e_2, e_3) = -\lambda_2 g(\nabla_{e_1} e_3, e_2) = \lambda_1 g(\nabla_{e_2} e_1, e_3) = -\lambda_1 g(\nabla_{e_2} e_3, e_1)$$
$$= (\lambda_1 - \lambda_2) g(\nabla_{e_3} e_1, e_2) = -(\lambda_1 - \lambda_2) g(\nabla_{e_3} e_2, e_1)\,.$$

Therefore, the covariant derivatives $\nabla_{e_r} e_s$ on W are given by

$$\nabla_{e_1} e_1 = \tfrac{e_2(\lambda_1)}{\lambda_1 - \lambda_2} e_2 + \alpha e_3,\ \nabla_{e_1} e_2 = -\tfrac{e_2(\lambda_1)}{\lambda_1 - \lambda_2} e_1 + \gamma e_3,\ \nabla_{e_1} e_3 = -\alpha e_1 - \gamma e_2\,,$$
$$\nabla_{e_2} e_1 = \tfrac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_2 + \beta e_3,\ \nabla_{e_2} e_2 = -\tfrac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_1 - \alpha e_3,\ \nabla_{e_2} e_3 = -\beta e_1 + \alpha e_2\,,$$
$$\nabla_{e_3} e_1 = \tfrac{\lambda_2 \gamma}{\lambda_1 - \lambda_2} e_2, \qquad \nabla_{e_3} e_2 = -\tfrac{\lambda_2 \gamma}{\lambda_1 - \lambda_2} e_1, \qquad \nabla_{e_3} e_3 = 0\,,$$

where we put $\alpha = g(\nabla_{e_1} e_1, e_3)$, $\beta = g(\nabla_{e_2} e_1, e_3)$ and $\gamma = g(\nabla_{e_1} e_2, e_3)$. Moreover, from the Codazzi equation as before explicited, we get that α, β and γ satisfy

$$\alpha = e_3(\lambda_1)/\lambda_1 = -e_3(\lambda_2)/\lambda_2, \qquad \lambda_1 \beta = \lambda_2 \gamma\,. \tag{11}$$

Moreover, $e_3(H) = e_3(\lambda_1 + \lambda_2 + \lambda_3) = \alpha(\lambda_1 - \lambda_2)$.

Using the previous derivatives, we can compute the components of the curvature tensor R with respect to $\{e_1, e_2, e_3\}$. In particular, we get

$$\begin{aligned} R_{1313} &= e_3(\alpha) - \alpha^2 - \frac{2\lambda_1 \beta\gamma}{\lambda_1 - \lambda_2}\,, \\ R_{2323} &= -e_3(\alpha) - \alpha^2 + \frac{2\lambda_2 \beta\gamma}{\lambda_1 - \lambda_2}\,. \end{aligned} \tag{12}$$

On the other hand, from the Gauss equation we get $R_{1313} = R_{2323} = \tilde{c}$. Therefore, (12) yields

$$\alpha^2 + \beta\gamma = -\tilde{c}, \qquad e_3(\alpha) = \frac{H\beta\gamma}{\lambda_1 - \lambda_2}\,. \tag{13}$$

Using (12) and (13), we get $e_3^2(H) = -\tilde{c}H$ and thus,

$$e_3^{2k}(H) = (-\tilde{c})^k H, e_3^{2k+1}(H) = (-\tilde{c})^k \alpha(\lambda_1 - \lambda_2)\,. \tag{14}$$

Since $e_3^k(H) = 0$ for some k and so, for all $e_3^r(H) = 0$ for all $r \geq k$. Then, (14) gives that $\alpha = 0$ and $H = 0$, that is, M is minimal. Moreover, $\lambda_2 = -\lambda_1$ and thus

(5) implies easily that λ_1 and λ_2 are constant on W and hence on M. Therefore, M is isoparametric and Proposition 1 implies that M is locally homogeneous.

Finally, taking into account the constancy of λ_1 and λ_2 and $\alpha = 0$, the previous formulas for the covariant derivatives give easily $R_{1212} = 2\beta\gamma$, while the Gauss equation gives $R_{1212} = \tilde{c} + \lambda_1\lambda_2$. From (13) we also have $\beta\gamma = -\tilde{c}$. Hence, $\lambda_1\lambda_2 = -3\tilde{c}$ and, since $\lambda_2 = -\lambda_1$, we get $\lambda_1^2 = 3\tilde{c}$. So, $\tilde{c} > 0$ and $\lambda_1 = -\lambda_2 = \sqrt{3\tilde{c}}$. □

Remark. E. Cartan proved in 1939 that minimal isoparametric hypersurfaces M^n of type 3, that is, having three distinct constant principal curvatures, only exist for $n = 3, 6, 12, 24$. Moreover, such a hypersurface is unique in each of these dimensions, up to a rotation of the sphere. These hypersurfaces are called *Cartan's minimal hypersurfaces* ([4], [15]). So, under the hypotheses of Theorem 3 we obtain a Cartan's minimal hypersurface.

Theorem 4. *A compact curvature homogeneous hypersurface M of the hyperbolic space $H^4(\tilde{c})$ is locally homogeneous.*

Proof. From the proof of Theorem 2 it follows that one has to consider only the case described by (10). Since $\lambda_3 = 0$ we get $\varrho_3 = 2\tilde{c} < 0$ and $\varrho_1 = \varrho_2$. Then Proposition 5.1 of [18] gives that M is locally homogeneous. □

Theorem 5. *A compact curvature homogeneous hypersurface M of the sphere $S^4(\tilde{c})$ having scalar curvature $\tau \geq 2\tilde{c}$ is locally homogeneous. More precisely,*

(i) *$M = S^3(\tilde{c})$, or*

(ii) *M is a Clifford torus, or*

(iii) *M is a non-minimal hypersurface.*

Proof. Using the proof of Theorem 2, we are left with the case given by (10). Then $\varrho_1 = \varrho_2 = \lambda_1\lambda_2 + 2\tilde{c}$ and $\varrho_3 = 2\tilde{c}$. Hence, $\tau = \varrho_1 + \varrho_2 + \varrho_3 = \lambda_1\lambda_2 + 6\tilde{c}$. By hypothesis, $\tau \geq 2\tilde{c}$, from which it follows $2\varrho_1 = \tau - \varrho_3 = \tau - 2\tilde{c} \geq 0$. Then Proposition 5.1 of [18] gives that M is locally homogeneous. In particular, if M is minimal, then from the classification theorem of [6] it follows that M is either an equatorial 3-sphere ($\tau = 6\tilde{c}$) or a Clifford torus ($\tau = 3\tilde{c}$). □

Theorem 6. *Let M be a curvature homogeneous hypersurface M of $S^4(\tilde{c})$ with scalar curvature $\tau \geq 6\tilde{c}$. Then M is locally homogeneous.*

Proof. If (10) is not satisfied then M is locally homogeneous. Suppose now (10) holds. With the same notations of the proof of Theorem 3, we have $\alpha^2 + \beta\gamma = -\tilde{c} < 0$, that is, $\beta\gamma = -\alpha^2 - \tilde{c} < 0$. But $\frac{\lambda_1}{\lambda_2} = \frac{\gamma}{\beta} < 0$ and hence $\lambda_1\lambda_2 < 0$. This implies $\tau = \lambda_1\lambda_2 + 6\tilde{c} < 6\tilde{c}$, which contradicts the hypothesis. □

A *ball-homogeneous space* is a Riemannian manifold such that the volume of "small" geodesic spheres or balls only depends on the radius and not on the center. Clearly, locally homogeneous spaces are ball-homogeneous while the converse is still an open problem, even in dimension 3 (we refer to [3] for a survey). Since ball-homogeneity and curvature homogeneity are both necessary conditions for local homogeneity, it is interesting to investigate whether they are taken together, also sufficient for local homogeneity or not. The following Theorem gives a positive answer in the case of hypersurfaces of a real space form.

Theorem 7. *A curvature homogeneous and ball-homogeneous hypersurface M of a four-dimensional space form $\tilde{M}^4(\tilde{c})$ is locally homogeneous.*

Proof. Using Theorem 2, the conclusion follows from the curvature homogeneity of M, except when $\varrho_1 = \varrho_2 \neq \varrho_3$. In this last case, since M is ball-homogeneous, it is also locally homogeneous, as it has been proved in Theorem 3.2 of [2]. □

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