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A VARIANT OF THE COMPLEX LIOUVILLE-GREEN APPROXIMATION THEOREM

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Abstract. We propose a variant of the classical Liouville-Green approximation theorem for linear *complex* differential equations of the second order. We obtain rigorous error bounds for the asymptotics *at infinity*, in the spirit of F. W. J. Olver's formulation, by using rather arbitrary ξ-progressive paths. This approach can provide higher flexibility in practical applications of the method.

Within the well-known Liouville-Green (or WKB) approximation theory for linear second-order differential equations, asymptotic results equipped with rigorous error bounds have been proved by F. W. J. Olver [1, Ch. 6], on both the *real* and the *complex* domain. The key theorem in the complex case, concerning asymptotics at infinity, can be formulated as

Theorem 1. *(Olver,* [1, Ch. 6, Thm. 11.1]*). Consider the differential equation*

$$y'' + [f(z) + g(z)]\, y = 0\,, \quad z \in \Omega\,, \tag{1}$$

$\Omega \subseteq \mathbf{C}$ being a simply connected unbounded region, where f and g are holomorphic, and $f(z) \neq 0$. Assume that there exist two subsets of Ω, H_1 and H_2, having the following properties:

(i) *for $j = 1, 2$, there is a family of paths, $\{\ell_j(z)\}$, indexed by $z \in H_j$, such that each $\ell_j(z)$ connects z to ∞ in Ω, $\ell_j(z)$ being a finite chain of regular arcs, and ξ-progressive, i.e., defining*

$$\xi(z) := \int f^{1/2}\, dz\,, \tag{2}$$

$Im\, \xi(z)$ turns out to be nondecreasing along $\ell_1(z)$ and nonincreasing along $\ell_2(z)$;

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(ii) *each* $\ell_j(z)$ *coincides with a given* z*-independent curve* $\mathcal{L}_j$ *in a neighborhood of* ∞*;*

(iii) *the variation* $V_{\ell_j(z)}(F)$ *of*

$$F(z) := \int \left\{ f^{-1/4} \frac{d^2}{dz^2}(f^{-1/4}) + f^{-1/2} g \right\} dz \tag{3}$$

on $\ell_j(z)$ *is finite for every* $z \in H_j$*. Here the branches of the fractional powers of* $f(z)$ *must be continuous in* Ω*, that of* $f^{1/2}(z)$ *being the square of* $f^{1/4}(z)$*.*

Then there exist two solutions of (1)

$$y_j(z) = f^{-1/4}(z) \exp\{i(-1)^{j-1}\xi(z)\}\,[1 + \varepsilon_j(z)]\,, \tag{4}$$

holomorphic in Ω*, with the error terms estimated as*

$$|\varepsilon_j(z)|, |f^{-1/2}(z)\varepsilon_j'(z)| \le \exp\{V_{\ell_j(z)}(F)\} - 1\,,\ z \in H_j\,, \quad j = 1, 2\,. \tag{5}$$

The curves $\ell_j(z)$ in (i) are required to satisfy two main properties, to be ξ-*progressive*, and to *coincide* with the fixed curve $\mathcal{L}_j$ in a neighborhood of infinity (in general depending on z). Both these properties represent an essential ingredient of Olver's proof. While the former seems to be difficult to weaken, being decisive in deriving the estimates (5), we propose to modify the latter, which plays a role mainly in proving uniqueness for the integral equations satisfied by the error terms (cf. [1, Ch. 6, §11.2]). Such a modification consists in allowing to connect z to infinity using rather arbitrary ξ-progressive paths in Ω (still satisfying (iii)), which fact confers a higher flexibility in practical applications of the Liouville-Green approximation, as it will be made clear in the examples below. Here is the variant of Olver's theorem, whose a preliminary version has been sketched in [5].

Theorem 2. *The thesis of Theorem 1 remains valid when* (ii) *is replaced with*

(ii') *if two paths of the family above,* $\ell_j(z)$ *and* $\ell_j(w)$*, meet at some point, then they must coincide from that point to* ∞*,*

and the following is required: H_1 *and* H_2 *have a nonempty interior, each path* $\ell_j(z)$ *lies in* H_j*, and the variation* $V_{\ell_j(z)}(F)$ *is bounded* (*as a function of* z) *on the compact subsets of* H_j°*,* $j = 1, 2$*.*

Before proving the theorem, some observations are in order. First of all, note that in (ii'), others than in (ii), only when two paths of the family $\{\ell_j(z)\}$ meet, then they must coincide from the merging point up to ∞, and so all paths of such a family do not need to lie asymptotically on a given fixed curve. In particular, condition (ii') implies that if $u \in \ell_j(z)$, then $\ell_j(u) \equiv \ell_j(z)$ from u up to ∞. In a (annular) sector, for instance, the bundle of rays could be a natural candidate as family of paths. Property (ii') plays a key role also in other asymptotic results for linear differential equations on the complex domain, cf. [3].

Differently from Olver's theorem, we require explicitly that H_j°, the interior of H_j, be nonempty. This property is verified in many applications. On the other hand, it can be guaranteed *a priori* when the monotonicity on the ξ-progressive paths is strict, and the ξ-map of $\ell_j(z)$ is a polygonal arc (we omit the proof for brevity). Indeed, the same is true for Olver's theorem, cf. Ex. 11.2 in [1, Ch. 6], which goes back to [4]. There, in addition, using the reference curve $\mathcal{L}_j$, connectdness of H_j can be immediately shown, while it is not guaranteed within the present formulation.

Finally, we observe that the "error control function", $V_{\ell_j(z)}(F)$, in general is not a continuous function of z (in both Theorem 1 and Theorem 2), and thus its boundedness on compact subsets of H_j° must be explicitly required. Boundedness is trivially verified, for instance, in the frequent case when $V_{\ell_j(z)}(F) = O(z^{-p})$, $p > 0$, and $z = 0$ is not in Ω. We stress, however, that in general Theorem 2 (as well as Theorem 1) only provides *pathwise* asymptotics at ∞. We do not enter into the delicate matter of *uniform* asymptotics. For a rather detailed discussion on the latter topics in a similar context, we refer to [3].

Proof of Theorem 2. Recall first that, for any given parametrization of the path $\ell_j(z)$, $\eta \equiv \eta_{j,z} : [t_z, t_\infty) \to H_j$, where $\eta(t_z) = z \in H_j$, $\eta(t_\infty) = \infty$, $j = 1, 2$, the variation of F on $\ell_j(z)$ is defined as

$$V_j(z) \equiv V_{\ell_j(z)}(F) := \int_{t_z}^{t_\infty} |F'(\eta(t))|\,|\eta'(t)|\,dt\,; \tag{6}$$

clearly, such a definition does not depend on the specific parametrization. Choosing $j = 1$ in (4) and inserting in (1), the "error equation",

$$\begin{aligned} \varepsilon_1'' &+ \left[-\frac{1}{2}f^{-1}f' + 2if^{1/2}\right]\varepsilon_1' \\ &+ \left[-\frac{1}{4}f^{-1}f'' + \frac{5}{16}(f^{-1}f')^2 + g\right](1+\varepsilon_1) = 0 \end{aligned} \tag{7}$$

is obtained by easy calculations. Then, the integral equation

$$\varepsilon_1(z) = -\frac{i}{2}\int_{\ell_1(z)} [1 - e^{-2i(\xi(z)-\xi(u))}]\,\psi(u)\,[1+\varepsilon_1(u)]\,du\,, \tag{8}$$

where

(9)

$$\psi := f^{-1/2}(z)\left[-\frac{1}{4}f^{-1}f'' + \frac{5}{16}(f^{-1}f')^2 + g\right] = f^{-1/4}(f^{-1/4})'' + f^{-1/2}g$$

is promptly derived (by the method of variation of parameters, [1, Ch. 6, §11.2]), whose holomorphic solutions also satisfy (7). Note that $\psi(z) \equiv F'(z)$ (cf. (3)). We now set

$$h_{s+1}(z) = -\frac{i}{2}\int_{\ell_1(z)} [1 - e^{-2i(\xi(z)-\xi(u))}]\,\psi(u)\,[1+h_s(u)]\,du\,, \qquad s = 0, 1, 2, \ldots, \tag{10}$$

where $h_0 \equiv 0$, and it is easy to show (by induction on s) that all functions $h_s(z)$ are well-defined and holomorphic in H_1°. Then, defining

$$\varepsilon_1(z) := \sum_{s=0}^{\infty} [h_{s+1}(z) - h_s(z)]\,, \tag{11}$$

we can derive the estimate (again by induction on s)

$$|h_{s+1}(z) - h_s(z)| \le \frac{[V_1(z)]^{s+1}}{(s+1)!}\,, \tag{12}$$

valid in H_1. In fact, the estimate holds trivially for $s = 0$; assuming that it holds for $s-1$, we obtain

$$|h_{s+1}(z) - h_s(z)| \le \int_{t_z}^{t_\infty} |\psi(\eta(t))|\,|\eta'(t)|\,\frac{[V_1(\eta(t))]^s}{s!}\,dt\,, \tag{13}$$

as $|\exp\{-2i[\xi(z) - \xi(\eta(t))]\}| \le 1$ for every $t \in [t_z, t_\infty)$, in view of the ξ-progressivity of $\ell_1(z)$. The result then follows from (6), since

$$V_1(\eta(t)) = \int_t^{t_\infty} |\psi(\eta(\tau))|\,|\eta'(\tau)|\,d\tau \quad t \in [t_z, t_\infty)\,, \tag{14}$$

owing to property (ii').

It follows that the series in (11) converges *pointwise* in H_1, and by the boundedness of V_1 it converges also *uniformly* on every compact subset of H_1°. Therefore $\varepsilon_1(z)$ is holomorphic in H_1° (by Weierstrass' theorem). It is easy to show (by the dominated convergence theorem) that $\varepsilon_1(z)$ solves the integral equation (8), and hence the error equation (7) in H_1°. From (11) and (12) the estimate for $\varepsilon_1(z)$ in (5) follows immediately. Finally, $y_1(z)$ solves the original equation (1) in H_1°, and hence it can be continued holomorphically into the whole Ω (which is simply connected). The estimate in (5) concerning $\varepsilon_1'(z)$ can be obtained differentiating in (11) and estimating $|h'_{s+1}(z) - h'_s(z)|$, proceeding as above. The part of the proof regarding $j = 2$ is completely analogous. $\square$

For the purpose of illustration, we work out in detail an application to the simple case of the "perturbed Airy equation",

$$y'' + [z + g(z)]y = 0\,, \quad g(z) = O(z^\alpha)\,, \quad \alpha < -1/2\,, \quad z \in \Omega\,,$$

$$\Omega = \{z \in \mathbf{C} : -\pi < \arg z < \pi,\ |z| > r \ge 0\}\,, \tag{15}$$

where the higher flexibility and the ensuing advantage of the present formulation from an *operative* standpoint will be shown. It is worth noting that the present variant has already been used in [2], as a matter of fact, in the special case when $f(z) \equiv a > 0$, and $g(z) = O(z^\alpha)$, $\alpha < -1$.

Now, choosing the principal branch of the powers involved, we have

$$F'(z) = \frac{5}{16} z^{-5/2} + z^{-1/2} g(z) = O\left(z^{-\beta}\right)\,, \quad \beta = \min\{5/2, -\alpha + 1/2\}\,. \tag{16}$$

When $F'(z) = O(z^{-\beta})$, with $\beta > 1$, and Ω is a (annular) sector, choosing *circular arcs* (centered at the origin) and *rays* to obtain paths on which the variation is finite, is particularly attractive since the whole construction and the calculations become extremely simple. On the other hand, being here $\xi(z) = \frac{2}{3} z^{3/2}$, it is immediately seen that such paths are ξ-progressive in suitable subsectors of Ω.

To be more precise, in applying Theorem 2, we divide Ω into the six pairwise disjoint (annular) subsectors

$$S_k = \left\{ z \in \Omega : -\pi + k\frac{\pi}{3} < \arg z \leq -\frac{2\pi}{3} + k\frac{\pi}{3} \right\}, \quad k = 0, \dots, 5\,. \tag{17}$$

Here, each ray separating any two of these subsectors, is either a Stokes line, i.e. a line where $\operatorname{Re}\xi(z) = 0$, or an anti-Stokes (also termed principal) line, where $\operatorname{Im}\xi(z) = 0$, cf. [1, Ch. 13]. Indeed, the relevant principal lines are given by $\arg z = -\frac{2\pi}{3} + j\frac{2\pi}{3}$, $j = 0, 1, 2$, and the Stokes lines by $\arg z = -\frac{\pi}{3}$ and $\arg z = \frac{\pi}{3}$ (the other one being the negative real axis, which has been deleted from Ω and hence from S_5). Considering sign and monotonicity of $\sin\left(\frac{3}{2}\arg z\right)$, it is easily seen that, in each S_k, one can choose as ξ-progressive paths the portion of ray joining z to ∞, and the circular arc (centered at the origin) from z to the principal line bordering S_k, followed by the line itself. For instance, in S_0, S_3, and S_4, $\ell_1(z)$ is the ray, and $\ell_2(z)$ is the circular arc followed by the principal line, while in the remaining subsectors the roles of the two paths have to be interchanged. Trivial calculations give the bound

$$V_j(z) \leq K_1 \left(\frac{\pi}{3} + \frac{1}{\beta - 1} \right) |z|^{1-\beta}, \quad j = 1, 2, \quad K_1 = K + \frac{5}{16}, \tag{18}$$

for every $z \in H_1 \cap H_2 = \Omega$; here K is the constant implied by the O-symbol in (15), cf. (16). The bound above provides immediately the asymptotic result $\varepsilon_j(z) = O(z^{1-\beta})$, $z \in \Omega$, where the constant implied can be easily estimated via (18) in any fixed neighborhood of ∞, exploiting the basic inequality $e^v - 1 \leq v\,e^v$, $v \geq 0$.

On the other hand, constructing the paths by means of arcs and rays in applying Theorem 1 to (15), causes a severe restriction of the subset $H_1 \cap H_2$. In fact, even with the best choice for the reference rays, $\mathcal{L}_j$, that is the Stokes lines $\arg z = -\frac{\pi}{3}$ and $\arg z = \frac{\pi}{3}$ as $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, Ω differs from $H_1 \cap H_2$ by the union of three subsectors of angle $\pi/3$. In other words, the estimates in (5) cannot be used simultaneously in an angular region with overall range of at least π; the easy but tedious check is left to the reader. This drawback could be circumvented by choosing different families of paths, at the price of a much more cumbersome analysis which goes through the ξ-plane, cf. [1, Ch. 13]).

The procedure sketched above can be extended to treat the perturbed "generalized" Airy equation

$$y'' + [z^m + g(z)]\,y = 0\,, \quad z \in \Omega\,,$$

$$\text{where } m \in \mathbf{N}\,, \quad g(z) = O(z^\alpha)\,, \quad \alpha < -m/2\,, \tag{19}$$

and Ω is as above. This embodies a wide class of equations with *polynomial* coefficients. Again, we easily get by Theorem 2 that $\varepsilon_j(z) = O(z^{1-\beta})$ in $H_1 \cap H_2 = \Omega$, $j = 1, 2$, where $\beta = \min\{m/2 + 2, m/2 - \alpha\}$. Clearly, the difficulties inherent Theorem 1 (using arcs and rays) still arise in the generalized Airy case; details are omitted for short. In closing, we stress that the main advantage of the variant in these applications is given, essentially, by the possibility of using *different* principal lines to construct ξ-progressive paths having the two types of monotonicity.

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