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SOME COMMON FIXED POINT THEOREMS FOR BIASED MAPPINGS

NASEER SHAHZAD AND SALMA SAHAR

Abstract. Some common fixed point theorems in normed spaces are proved using the concept of biased mappings- a generalization of compatible mappings.

1. Introduction and preliminaries

Jungck [2] generalized the concept of commuting mappings by introducing compatible mappings. Several authors proved common fixed point theorems using this concept (see, for example, the work of Pathak and Fisher [6], Jungck [3], and Kaneko and Sessa [1]). Jungck, Murthy and Cho [4] gave the notion of compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions. Afterwards, Pathak and Khan [8] introduced the concept of compatible mappings of type (B) and compared these mappings with compatible mappings and compatible mappings of type (A) in normed spaces. A related but different concept was also given by Pathak, Kang and Cho [7]. Recently, Jungck and Pathak [5] introduced a generalization of compatible mappings called "biased mappings". The purpose of this paper is to prove some common fixed point theorem using this concept. We also generalize a recent result of Pathak and Fisher [6]. It is worth mentioning that the class of biased maps includes the class of compatible maps and commuting maps as well.

Let (X,d) be a metric space. The self-mappings A and $B : X \to X$ are said to be compatible if

$$\lim_n d(ABx_n, BAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Bx_n = t$, for some $t \in X$. The pair $\{A, B\}$ is B-biased iff whenever $\{x_n\}$ is a sequence in X and $Ax_n,\ Bx_n \to t \in X$, then

$$\alpha d(BAx_n, Bx_n) \le \alpha d(ABx_n, Ax_n)$$

if $\alpha = \liminf$ and if $\alpha = \limsup$.

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If the pair $\{A, B\}$ is compatible, then it is both A- and B-biased [5]. However, the converse is not true in general.

Example [5]. Let $X = [0, 1]$ with the usual metric d. Define the mappings $A, B : X \to X$ by

$$Ax = \begin{cases} 1 - 2x & \text{if } x \in [0, 1/2] \\ 0 & \text{if } x \in (1/2, 1] \end{cases}$$

and

$$Bx = \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 1 & \text{if } x \in (1/2, 1] \,. \end{cases}$$

Then the pair $\{A, B\}$ is both A- and B-biased but not compatible.

The pair $\{A, B\}$ is weakly B-biased iff $Ap = Bp$ implies $d(BAp, Bp) \leq d(ABp, Ap)$. For more details, we refer to Jungck and Pathak [5].

2. Main Results

Theorem 2.1. *Let A and B be two self-mappings of a normed space X and let C be a closed, convex and bounded subset of X satisfying the following condition.*

$$\begin{aligned} ||Ax - Ay||^p &\leq a||Bx - By||^p + (1-a) \\ &\times \max\left\{\frac{||Ax - By||^p}{2}, \frac{||Ay - Bx||^p}{2}\right\}, \end{aligned} \tag{1}$$

$$B(C) \supseteq (1-k)\, B(C) + kA(C) \tag{2}$$

for all $x, y \in C$, where $0 < a < 1$, $p > 0$, and for some fixed k such that $0 < k < 1$. Suppose, for some $x_0 \in C$, the sequence $\{x_n\} \subset X$ defined inductively for $n = 0, 1, 2, \ldots$ by

$$Bx_{n+1} = (1-k)Bx_n + kAx_n \tag{3}$$

converges to a point z of C and the pair $\{A, B\}$ is a B-biased. If B is continuous at z, then A and B have a unique common fixed point. Further, if B is continuous at Az, then A and B have a unique common fixed point at which A is continuous.

Proof. First, we are going to prove that $Az = Bz$.

We have

$$\begin{aligned} ||Bz - Az||^p &= ||Bz - Bx_{n+1} + Bx_{n+1} - Az||^p \\ &\leq (||Bz - Bx_{n+1}|| + ||Bx_{n+1} - Az||)^p \,. \end{aligned} \tag{4}$$

Now, from (3), we obtain

$$\begin{aligned} ||Bx_{n+1} - Az||^p &= ||(1-k)Bx_n + kAx_n - Az||^p \\ &= ||(1-k)(Bx_n - Az) + k(Ax_n - Az)||^p \\ &\leq ((1-k)||Bx_n - Az|| + k||Ax_n - Az||)^p \end{aligned}$$

and so

$$\begin{aligned}\|Bx_{n+1}-Az\|^p &\le \big[(1-k)\|Bx_n-Az\| \\ &\qquad + k(\|Ax_n-Az\|^p)^{1/p}\big]^p .\end{aligned}\tag{5}$$

It follows, from (1), that

$$\begin{aligned}\|Ax_n-Az\|^p &\le a\|Bx_n-Bz\|^p+(1-a)\\ &\quad\times\max\left\{\frac{\|Ax_n-Bx\|^p}{2},\frac{\|Az-Bx_n\|^p}{2}\right\}.\end{aligned}$$

Now, since B is continuous at z, it follows that $Bx_n\to Bz$ as $n\to\infty$. Also, from (3), we have

$$\|Ax_n-Bz\|\to 0 \quad\text{as}\quad n\to\infty .$$

Therefore, for every $\varepsilon>0$ and sufficiently large n,

$$\|Ax_n-Az\|^p\le\frac{(1-a)\|Az-Bz\|^p}{2}+\varepsilon .\tag{6}$$

Hence, from (4), (5) and (6), it follows that

$$\|Bz-Az\|^p<\left[(1-k)+k\frac{(1-a)^{1/p}}{2}\right]^p\|Bz-Az\|^p ,$$

which is a contradiction. Therefore, $Bz=Az$.

Let $w=Az=Bz$. Since $\{A,B\}$ is B-biased, it is weakly B-biased. It implies that

$$\|BAz-Bz\|\le\|ABz-Az\| ,$$

that is

$$\|Bw-w\|\le\|Aw-w\| .\tag{7}$$

We assert that $Aw=w$. If not, then

$$\begin{aligned}\|Aw-Ax_{n+1}\|^p &\le a\|Bw-Bx_{n+1}\|^p+(1-a)\\ &\quad\times\max\left\{\frac{\|Ax_{n+1}-Bw\|^p}{2},\frac{\|Aw-Bx_{n+1}\|^p}{2}\right\}.\end{aligned}$$

Letting $n\to\infty$, we get

$$\begin{aligned}\|Aw-w\|^p &\le a\|Bw-w\|^p+(1-a)\\ &\quad\times\max\left\{\frac{\|w-Bw\|^p}{2},\frac{\|Aw-w\|^p}{2}\right\}\\ &\le\frac{(1+a)}{2}\|Aw-w\|^p ,\end{aligned}$$

which is a contradiction, since $0 < a < 1$. Thus, $Aw = w$. It follows from (7) that $Bw = w$. Hence $w = Aw = Bw$, that is, $w = Az$ is a common fixed point of A and B.

Now, let $\{y_n\}$ be a sequence in C with the limit $Az = w$. Then using the condition (1), we obtain

$$\begin{aligned}||Ay_n - Aw||^p &\leq a||By_n - Bw||^p + (1-a)\\ &\quad \times \max\left\{\frac{||Ay_n - Bw||^p}{2}, \frac{||Aw - By_n||^p}{2}\right\}.\end{aligned}$$

Since B is continuous at $Az = w$, we have for sufficiently large n and $\varepsilon > 0$

$$||Ay_n - Aw||^p \leq \frac{(1-a)}{2}||Ay_n - Bw||^p + \varepsilon\,.$$

Again, since

$$w = Bw = Aw$$

we have, for sufficiently large n and $\varepsilon > 0$

$$||Ay_n - Aw||^p \leq \frac{(1-a)}{2}||Ay_n - Aw||^p + \varepsilon\,,$$

that is

$$\lim_n ||Ay_n - Aw|| = 0\,,$$

which means that A is continuous at Az.

Let w and w_1 be two common fixed point of A and B. Then

$$w = Aw = Bw \tag{8}$$

and

$$w_1 = Aw_1 = Bw_1\,. \tag{9}$$

It follows, from (1), that

$$\begin{aligned}||Aw - Aw_1||^p &\leq a||Bw - Bw_1||^p + (1-a)\\ &\quad \times \max\left\{\frac{||Aw - Bw_1||^p}{2}, \frac{||Aw_1 - Bw||^p}{2}\right\}.\end{aligned} \tag{10}$$

From (8), (9) and (10), it follows that

$$||Bw - Bw_1||^p \leq \frac{(a+1)}{2}||Bw - Bw_1||^p\,,$$

which is a contradiction. Therefore,

$$w = Bw = Bw_1 = Aw = Aw_1 = w_1\,.$$

This completes the proof. $\square$

Corollary 2.2. *Let A be a mapping of a normed space X into itself and let C be a closed, convex and bounded subset of X satisfying the following condition:*

$$\begin{aligned}||Ax - Ay||^p \leq a||x-y||^p &+ (1-a) \\ &\times \max\left\{\frac{||Ax-y||^p}{2}, \frac{||Ay-x||^p}{2}\right\},\end{aligned}$$

and $C \supseteq (1-k)C + kA(C)$ for all x, y in C, where $0 < a < 1$ and $p > 0$, and for a fixed k such that $0 < k < 1$. If, for some $x_0 \in C$, the sequence $\{x_n\}$ in X inductively defined for $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = (1-k)x_n + kAx_n$$

converges to a point z of C, then A has a unique fixed point at which A is continuous.

Example 2.3. Let $X = [0, \infty)$ with the Euclidean norm and $C = [0, 1]$. Let A and B be self-mappings of X defined by

$$Ax = \begin{cases} 1 & \text{if } x \in [0,1]\,, \\ 1 + x^2 & \text{if } x \in (1, \infty) \end{cases}$$

and

$$Bx = \begin{cases} 1 + x^2 & \text{if } x \in [0,1)\,, \\ 1 & \text{if } x \in [1, \infty)\,. \end{cases}$$

For a fixed k such that $0 < k < 1$ we have

$$[1, 2) = B(C) \supseteq (1-k)\,B(C) + kA(C) = [1, 2-k)$$

and

$$||Ax - Ay||^p = 0$$

for all x, $y \in C$ and $p > 0$.

Consider a sequence $\{x_n\}$ in X. If Bx_n, $Ax_n \to t(=1) \in X$, then $x_n \to 0$. It follows that $||BAx_n - Bx_n|| \to 0$ and so $\{A, B\}$ is B-biased. Also, for any $x_0 \in C$, the sequence $\{x_n\}$ such that $Bx_{n+1} = (1-k)Bx_n + kAx_n$ for $n \geq 0$ converges to the point $z = 1$. Clearly, $A(1) = 1$ is a common fixed point of A and B.

Example 2.4. Let $X = [0, \infty)$ with the Euclidean norm and $C = [0, 1]$. Let A and B be self-mappings of X defined by

$$Ax = 1$$

and

$$Bx = \begin{cases} 1 + x & \text{if } x \in [0,1]\,, \\ 1 & \text{if } x \in (1, \infty)\,. \end{cases}$$

Then

$$\|Ax - Ay\|^p = 0$$

for all x, y in C and for all a, $0 < a < 1$ and $p > 0$. Also

$$\begin{aligned} B(C) &= [1,2] \supset [1, 2-k] \\ &= (1-k)\,B(C) + kA(C)\,. \end{aligned}$$

Consider a sequence $\{x_n\}$ in X converging to 0. Then Bx_n, $Ax_n \to t \in X$, but $\|BAx_n - Bx_n\| \to 1$ and $\|ABx_n - Ax_n\| \to 0$, as $x_n \to 0$. Consequently, $\{A, B\}$ is not B-biased. On the other hand, A and B do not have common fixed points.

Lemma 2.5. *Let A, B, S and T be self-mappings of a metric space (X, d). Suppose that*

$$d^p(Sx, Ty) \le \phi\left(\frac{ad^{2p}(Ax, By) + (1-a)\max\{d^{2p}(Sx, By), d^{2p}(Ty, Ax)\}}{\max\{d^p(Sx, By), d^p(Ty, Ax)\}}\right)$$

for all $x, y \in X$ for which $\max\{d^p(Sx, By), d^p(Ty, Ax)\} \neq 0$, where $0 < a < 1$, $p > 0$ and ϕ is a function which is upper semicontinuous from $\mathbb{R}^+$ into itself such that $\phi(t) < t$ for each $t > 0$. If there exists $u, v, w \in X$ such that

$$w = Su = Au = Tv = Bv\,,$$

and $\{A, S\}$ is weakly A-biased and $\{B, T\}$ is weakly B-biased, then

$$w = Sw = Aw = Tw = Bw\,.$$

Proof. Since $\{A, S\}$ is weakly A-biased,

$$d(ASu, Au) \le d(SAu, Su)\,,$$

that is

$$d(Aw, w) \le d(Sw, w)\,.$$

We assert that $Sw = w$, and hence $Aw = w$. If not, then $Sw \neq Bw$, and therefore

$$\max\{d^p(Sw, Bv), d^p(Tv, Aw)\} \neq 0$$

and so

$$\begin{aligned} d^p(Sw, w) &= d^p(Sw, Tv) \\ &\le \phi\left(\frac{ad^{2p}(Aw, Bv) + (1-a)\max\{d^{2p}(Sw, Bv), d^{2p}(Tv, Aw)\}}{\max\{d^p(Sw, Bv), d^p(Tv, Aw)\}}\right) \\ &= \phi\left(\frac{ad^{2p}(Aw, w) + (1-a)\max\{d^{2p}(Sw, w), d^{2p}(w, Aw)\}}{\max\{d^p(Sw, w), d^p(w, Aw)\}}\right) \\ &\le \phi\left(\frac{ad^{2p}(Sw, w) + (1-a)d^{2p}(Sw, w)}{d^p(Sw, w)}\right) < d^p(Sw, w)\,, \end{aligned}$$

a contradiction. Hence $Sw = w$ and so $Aw = w$. Similarly, we can prove that $Tw = Bw = w$. □

Theorem 2.6. *Let A, B, S and T be self-mappings of a normed space X. Let C be a closed convex subset of X such that*

$$(1-k)\,A(C) + kS(C) \subseteq A(C)\,, \tag{11}$$

$$(1-k')\,B(C) + k'T(C) \subseteq B(C)\,, \tag{12}$$

where $0 < k,\ k' < 1$ and suppose that

$$\begin{aligned}(13)\quad & \|Sx - Ty\|^p \\ & \le \phi\left(\frac{a\|Ax - By\|^{2p} + (1-a)\max\{\|Sx - By\|^{2p}, \|Ty - Ax\|^{2p}\}}{\max\{\|Sx - By\|^p, \|Ty - Ax\|^p\}}\right)\end{aligned}$$

for all $x, y \in C$ for which

$$\max\{\|Sx - By\|^p, \|Ty - Ax\|^p\} \neq 0\,,$$

where $0 < a < 1$, $p > 0$ and ϕ is a function which is upper semicontinuous from $\mathbb{R}^+$ into itself such that $\phi(t) < t$ for each $t > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2, 3, \ldots$ by

$$\begin{aligned} Ax_{2n+1} &= (1-k)Ax_{2n} + kSx_{2n}\,, \\ Bx_{2n+2} &= (1-k')Bx_{2n+1} + k'Tx_{2n+1} \end{aligned} \tag{14}$$

converges to a point $z \in C$, if A and B are continuous at z, and if $\{A, S\}$ is A-biased, $\{B, T\}$ is B-biased, then A, B, S and T have a unique common fixed point $w = Tz$ in C. Further, if A and B are continuous at w, then S and T are continuous at w.

Proof. First, we prove that

$$Az = Bz = Sz = Tz\,. \tag{15}$$

It follows, form (14), that

$$kSx_{2n} = Ax_{2n+1} - (1-k)Ax_{2n}\,,$$

and since A is continuous at z,

$$\lim_n Ax_n = \lim_n Sx_{2n} = Az\,.$$

Similarly,

$$\lim_n Bx_n = \lim_n Tx_{2n+1} = Bz\,.$$

Suppose that $Az \neq Bz$ such that for large enough n, $Sx_{2n} \neq Bx_{2n+1}$. Then, using (13) we obtain

$$\|Sx_{2n} - Tx_{2n+1}\|^p \leq \phi\left(\frac{a\|Ax_{2n} - Bx_{2n+1}\|^{2p} + (1-a)\max\{\|Sx_{2n} - Bx_{2n+1}\|^{2p}, \|Tx_{2n+1} - Ax_{2n}\|^{2p}\}}{\max\{\|Sx_{2n} - Bx_{2n+1}\|^p, \|Tx_{2n+1} - Ax_{2n}\|^p\}}\right).$$

Letting $n \to \infty$, it follows that

$$\begin{aligned}\|Az - Bz\|^p &\leq \phi\left(\frac{a\|Az - Bz\|^{2p} + (1-a)\max\{\|Az - Bz\|^{2p}, \|Bz - Az\|^{2p}\}}{\max\{\|Az - Bz\|^p, \|Bz - Az\|^p\}}\right)\\ &= \phi(\|Az - Bz\|^p) < \|Az - Bz\|^p\,,\end{aligned}$$

a contradiction. Therefore, $Az = Bz$.

Now suppose that $Tz \neq Az$ such that for large enough n, $Tz \neq Ax_{2n}$. Then, using (13) again, we obtain

$$\|Sx_{2n} - Tz\|^p \leq \phi\left(\frac{a\|Ax_{2n} - Bz\|^{2p} + (1-a)\max\{\|Sx_{2n} - Bz\|^{2p}, \|Tz - Ax_{2n}\|^{2p}}{\max\{\|Sx_{2n} - Bz\|^p, \|Tz - Ax_{2n}\|^p\}}\right).$$

Letting $n \to \infty$, we get

$$\begin{aligned}\|Az - Tz\|^p &\leq \phi\left(\frac{a\|Az - Bz\|^{2p} + (1-a)\max\{\|Az - Bz\|^{2p}, \|Tz - Az\|^{2p}\}}{\max\{\|Az - Bz\|^p, \|Tz - Az\|^p\}}\right)\\ &= \phi((1-a)\|Az - Tz\|^p) < (1-a)\|Az - Tz\|^p\,,\end{aligned}$$

a contradiction. Therefore, $Az = Tz$. Similarly $Sz = Bz$. Hence

$$Az = Bz = Sz = Tz\,.$$

Let

$$w = Az = Bz = Sz = Tz\,.$$

Then, by Lemma 2.5, we get

$$w = Aw = Bw = Sw = Tw\,.$$

Let $\{y_n\}$ be an arbitrary sequence in C converging to w and suppose that the sequence $\{Sy_n\}$ does not converge to Sw. Then, for large enough n, and using (13), we obtain

$$\begin{aligned}||Sy_n - Sw||^p &= ||Sy_n - Tw||^p \\ &\leq \phi\left(\frac{a||Bw - Ay_n||^{2p} + (1-a)\max\{||Sy_n - Bw||^{2p}, ||Tw - Ay_n||^{2p}\}}{\max\{||Sy_n - Bw||^p, ||Tw - Ay_n||^p\}}\right).\end{aligned}$$

Since A and B are continuous at w, it implies that, for arbitrary $\varepsilon > 0$ and sufficiently large n

$$\begin{aligned}||Sy_n - Sw||^p &\leq \phi((1-a)||Sy_n - Sw||^p + \varepsilon) \\ &< (1-a)||Sy_n - Sw||^p + \varepsilon,\end{aligned}$$

a contradiction since $a < 1$. Thus the sequence $\{Sy_n\}$ converges to Sw. Similarly, we can prove that T is also continuous at w. The uniqueness of the common fixed point follows from inequality (13). If w, w' are two common fixed points of A, B, S and T. Then

$$w = Aw = Bw = Sw = Tw$$

and

$$w' = Aw' = Bw' = Sw' = Tw'.$$

Now

$$\begin{aligned}||w-w'||^p &= ||Sw - Tw'||^p \\ &\leq \phi\left(\frac{a||Aw - Bw'||^{2p} + (1-a)\max\{||Sw - Bw'||^{2p}, ||Tw' - Aw||^{2p}\}}{\max\{||Sw - Bw'||^p, ||Bw' - Aw||^p\}}\right) \\ &= \phi\left(\frac{a||w - w'||^{2p} + (1-a)\max\{||w - w'||^{2p}, ||w' - w||^{2p}\}}{\max\{||w - w'||^p, ||w' - w||^p\}}\right) \\ &= \phi(||w - w'||^p) < ||w - w'||^p.\end{aligned}$$

This completes the proof. □

When $S = T$ and $A = B$, we have the following corollary:

Corollary 2.7. *Let A and S be self-mappings of a normed space X. Let C be a closed convex subset of X such that*

$$(1-k)\,A(C) + kS(C) \subseteq A(C),$$

where $0 < k < 1$ and suppose that

$$||Sx - Sy||^p \leq \phi\left(\frac{a||Ax - Ay||^{2p} + (1-a)\max\{||Sx - Ay||^{2p}, ||Sy - Ax||^{2p}\}}{\max\{||Sx - Ay||^p, ||Sy - Ax||^p\}}\right)$$

for all $x,\ y \in C$ *for which* $\max\{\|Sx - Ay\|^p,\ \|Sy - Ax\|^p \neq 0\}$, *where* $0 < a < 1$, $p > 0$ *and* ϕ *is a function which is upper semicontinuous from* $\mathbb{R}^+$ *into itself such that* $\phi(t) < t$ *for each* $t > 0$. *If, for some* $x_0 \in C$, *the sequence* $\{x_n\}$ *in* X *defined inductively for* $n = 0, 1, 2, 3, \ldots$ *by*

$$Ax_{n+1} = (1-k)Ax_n + kSx_n$$

converges to a point $z \in C$, *if* A *is continuous at* z, *and* $\{A, S\}$ *is* A*-biased, then* A *and* S *have a unique common fixed point* $Sz = w$ *in* C. *Further, if* A *is continuous at* w, *then* S *is continuous at* w.

When $A = B = I$, the identity mapping on X, we have the following corollary:

Corollary 2.8. *Let* S *and* T *be self-mappings of a normed space* X. *Let* C *be a closed convex subset of* X *such that*

$$\begin{aligned} &(1-k)C + kS(C) \subseteq C\,,\\ &(1-k)C + kT(C) \subseteq C\,, \end{aligned} \tag{16}$$

where $0 < k,\ k' < 1$ *and suppose that*

$$\|Sx - Ty\|^p \leq \phi\left(\frac{a\|x-y\|^{2p} + (1-a)\max\{\|Sx-y\|^{2p}, \|Ty-x\|^{2p}\}}{\max\{\|Sx-y\|^p, \|Ty-x\|^p\}}\right)$$

for all $x,\ y \in C$ *for which*

$$\max\{\|Sx-y\|^p, \|Ty-x\|^p\} \neq 0\,,$$

where $0 < a < 1$, $p > 0$ *and* ϕ *is a function which is upper semicontinuous from* $\mathbb{R}^+$ *into itself such that* $\phi(t) < t$ *for each* $t > 0$. *If, for some* $x_0 \in C$, *the sequence* $\{x_n\}$ *in* X *defined inductively for* $n = 0, 1, 2, 3, \ldots$ *by*

$$\begin{aligned} x_{2n+1} &= (1-k)x_{2n} + kSx_{2n}\,,\\ x_{2n+2} &= (1-k')x_{2n+1} + k'Tx_{2n+1} \end{aligned} \tag{17}$$

converges to a point $z \in C$, *then* T *and* S *have a unique common fixed point* $w = Tz$ *in* C. *Further,* T *and* S *are continuous at* w.

When $A = B = I$, the identity mapping and $\phi(t) = \alpha t$ for all $t > 0$ and $0 < \alpha < 1$, we have the following corollary:

Corollary 2.9. *Let* S *and* T *be self-mappings of a normed space* X. *Let* C *be a closed convex subset of* X *satisfying* (16) *and suppose that*

$$\|Sx - Ty\|^p \leq \alpha\left(\frac{a\|x-y\|^{2p} + (1-a)\max\{\|Sx-y\|^{2p}, \|Ty-x\|^{2p}\}}{\max\{\|Sx-y\|^p, \|Ty-x\|^p\}}\right)$$

for all $x,\ y \in C$ *for which*

$$\max\{\|Sx-y\|^p, \|Ty-x\|^p\} \neq 0\,,$$

where $0 < \alpha,\ a < 1$ *and* $p > 0$. *If, for some* $x_0 \in C$, *the sequence* $\{x_n\}$ *in* X *defined by (17) converges to a point* $z \in C$, *then* S *and* T *have a unique common fixed point* $w = Tz$ *in* C. *Further,* S *and* T *are continuous at* Tz.

Example 2.10. Let $X = [0, \infty)$ with the Euclidean norm and let $C = [0, 1]$. Define the mappings A, B, S and T of X into itself by

$$Ax = \begin{cases} 1 & \text{if } x \in [0, 1/2)\,, \\ x & \text{if } x \in [1/2, \infty)\,, \end{cases}$$

$$Sx = \begin{cases} 1 & \text{if } x \in [0, 1]\,, \\ 1 + x^2 & \text{if } x \in (1, \infty)\,, \end{cases}$$

$$Bx = \begin{cases} 1 & \text{if } x \in [0, 1/2)\,, \\ x^2 & \text{if } x \in [1/2, \infty) \end{cases}$$

and

$$Tx = \begin{cases} 1 & \text{if} x \in [0, 1]\,, \\ 1 + x^3 & \text{if } x \in (1, \infty)\,. \end{cases}$$

Then A and B are not continuous at $1/2$ and S and T are not continuous at 1. Consider a sequence $\{x_n\}$ such that

$$\lim_n Ax_n = \lim_n Sx_n = t\,.$$

Then $\lim_n \|ASx_n - SAx_n\| = 0$. Thus $\{S, A\}$ is compatible, and hence is both S and A-biased. Similarly, $\{B, T\}$ is both B and T-biased.

For fixed $k, k' \in (0, 1)$, we have

$$(1 - k)A(C) + kS(C) = [1/2 + 1/2k, 1] \subseteq A(C) = [1/2, 1]\,,$$

$$(1 - k')B(C) + k'T(C) = [1/4 + 3/4k', 1] \subseteq B(C) = [1/4, 1]$$

and

$$\|Sx - Ty\|^p = 0$$

for all $x, y \in C$ and $p > 0$. Also, for any $x_0 \in C$, the sequence $\{x_n\}$ in C such that

$$Ax_{2n+1} = (1 - k)Ax_{2n} + kSx_{2n}\,,$$
$$Bx_{2n+2} = (1 - k')Bx_{2n+1} + k'Tx_{2n+1}$$

for $n = 0, 1, 2, 3, \ldots$ converges to the point $z = 1$. Clearly, $w = T1$ is a common fixed point of A, B, S and T. For details, we refer to [6].

Example 2.11. Let $X = [0, \infty)$ with the Euclidean norm and let $C = [0, 1]$. Define the mappings B and T of X into itself by

$$Bx = \begin{cases} 1 + (1/2)x & \text{if } x \in [0,1]\,, \\ 1 & \text{if } x \in (1, \infty)\,, \end{cases}$$

$$Tx = 1$$

Then we see that $||Tx - Ty||^p = 0$ for all $x, y \in C$ with $p > 0$.

For some $k \in (0, 1)$, we have

$$(1-k)B(C) + kT(C) = [1, 3/2 - 1/2k] \subset B(C) = [1, 3/2]\,.$$

Also, if $\{x_n\}$ is a sequence in X converging to 0, then

$$\lim_n Bx_n = \lim_n Tx_n = 1\,,$$

but

$$\lim_n ||BTx_n - Bx_n|| = 1/2$$

and

$$\lim_n ||TBx_n - Tx_n|| = 0\,.$$

Consequently, $\{B, T\}$ is not B-biased. Clearly, B and T have no common fixed point in C. For details, see [6].

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N. Shahzad, Department of Mathematics
King Abdul Aziz University, Faculty of Science
P.O.Box 9028, Jeddah, 21413, SAUDI ARABIA
E-mail: naseer_shahzad@hotmail.com

S. Sahar, Department of Mathematics
Quaid-I-Azam University
Islamabad, PAKISTAN