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# ON LIE IDEALS AND JORDAN LEFT DERIVATIONS OF PRIME RINGS

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Abstract. Let $R$ be a 2-torsion free prime ring and let $U$ be a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. In the present paper it is shown that if $d$ is an additive mappings of $R$ into itself satisfying $d(u^2) = 2ud(u)$ for all $u \in U$, then $d(uv) = ud(v) + vd(u)$ for all $u, v \in U$.

## 1. Introduction

Throughout the present paper $R$ will denote an associative ring with centre $Z$. Recall that $R$ is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. As usual $[x, y]$ will denote the commutator $xy - yx$. An additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$ if $[u, r] \in U$ for all $u \in U$, $r \in R$. An additive mapping $d : R \longrightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$, (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Obviously every derivation is a Jordan derivation. The converse need not be true in general. A famous result due to Herstein [8] states that every Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of this result is presented in [4]. Further, Awtar [1] generalized this result on Lie ideals.

An additive mapping $d : R \longrightarrow R$ is called a left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in R$. Clearly, every left derivation is a Jordan left derivation. Thus, it is natural to question that: Whether every Jordan left derivation on a ring is a left derivation? In the present paper we have shown that the answer to the above question is affirmative in the case when the underlying ring $R$ is 2-torsion free and prime. In fact we have obtained rather a more general result which establish that under appropriate restriction on a Lie ideal $U$ of a 2-torsion free prime ring, every Jordan left derivation on $U$ is a left derivation on $U$.

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## 2. Preliminary results

We begin with the following results which will be used extensively to prove our theorem. Lemma 2.1 can be found in [2].

**Lemma 2.1.** *If $U \not\subseteq Z$ is a Lie ideal of a 2-torsion free prime ring $R$ and $a, b \in R$ such that $aUb = 0$, then $a = 0$ or $b = 0$.*

**Lemma 2.2.** *Let $R$ be a 2-torsion free ring and let $U$ be a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. If $d : R \longrightarrow R$ is an additive mapping satisfying $d(u^2) = 2ud(u)$ for all $u \in U$, then*

(i) $d(uv + vu) = 2ud(v) + 2vd(u)$, *for all* $u, v \in U$.
(ii) $d(uvu) = u^2d(v) + 3uvd(u) - vud(u)$, *for all* $u, v \in U$.
(iii) $d(uvw + wvu) = (uw + wu)d(v) + 3uvd(w) + 3wvd(u) - vud(w) - vwd(u)$, *for all* $u, v, w \in U$.
(iv) $[u, v]ud(u) = u[u, v]d(u)$, *for all* $u, v \in U$.
(v) $[u, v]\,(d(uv) - ud(v) - vd(u)) = 0$, *for all* $u, v \in U$.

**Proof.** (i) Since $uv + vu = (u + v)^2 - u^2 - v^2$, we find that $uv + vu \in U$ for all $u, v \in U$. Hence our hypothesis yields the required result.

(ii) Since $uv + vu \in U$, replacing $v$ by $uv + vu$ in (i), we get

$$d\left(u(uv + vu) + (uv + vu)u\right) = 4u^2d(v) + 6uvd(u) + 2vud(u) \tag{2.1}$$

On the other hand

$$\begin{aligned} d\left(u(uv + vu) + (uv + vu)u\right) &= d(u^2v + vu^2) + 2d(uvu) \\ &= 2u^2d(v) + 4vud(u) + 2d(uvu) \end{aligned}$$

Combining the above equation with (2.1) we get (ii)

(iii) By linearizing (ii) on $u$, we get

$$\begin{aligned} d\left((u + w)v(u + w)\right) = {} & u^2d(v) + w^2d(v) + (uw + wu)d(v) \\ & + 3uvd(u) + 3uvd(w) + 3wvd(u) \\ & + 3wvd(w) - vud(u) - vud(w) \\ & - vwd(u) - vwd(w) \end{aligned} \tag{2.2}$$

On the other hand

$$\begin{aligned} d\left((u + w)v(u + w)\right) &= d(uvu) + d(wvw) + d(uvw + wvu) \\ &= u^2d(v) + 3uvd(u) - vud(u) + w^2d(v) \\ &\quad + 3wvd(w) - vwd(w) + d(uvw + wvu) \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we get the result.

(iv) Since $uv + vu$ and $uv - vu$ both belong to $U$, we find that $2uv \in U$ for all $u, v \in U$. Hence, by our hypothesis we find that $d(uv)^2 = 2uvd(uv)$. Replace $w$

by $2uv$ in (iii), and use the fact that char $R \neq 2$, to get

$$\begin{aligned} d\left(uv(uv) + (uv)vu\right) &= (u^2v + uvu)d(v) + 3uvd(uv) \\ &\quad + 3uv^2d(u) - vud(uv) - vuvd(u) \end{aligned} \tag{2.4}$$

On the other hand

$$\begin{aligned} d\left((uv)uv + (uv)vu\right) &= d((uv)^2 + uv^2u) \\ &= 2uvd(uv) + 2u^2vd(v) + 3uv^2d(u) - v^2ud(u) \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$[u,v]d(uv) \;=\; u[u,v]d(v) + v[u,v]d(u)\,, \quad \text{for all } u,v \in U\,. \tag{2.6}$$

Replacing $u+v$ for $v$ in (2.6), we have

$$2[u,v]ud(u) + [u,v]d(uv) \;=\; 2u[u,v]d(u) + u[u,v]d(v) + v[u,v]d(u)\,.$$

Now application of (2.6) yields the required result.

(v) Linearize (iv) on $u$, to get

$$\begin{aligned} &[u,v]ud(u) + [u,v]vd(v) + [u,v]ud(v) + [u,v]vd(u) \\ &\quad = u[u,v]d(u) + u[u,v]d(v) + v[u,v]d(u) + v[u,v]d(v)\,, \quad \text{for all } u,v \in U\,. \end{aligned}$$

Application of (iv) and (2.6) yield that $[u,v]ud(v) + [u,v]vd(u) \;=\; [u,v]d(uv)$ and hence $[u,v]\{d(uv) - ud(v) - vd(u)\} = 0$, for all $u,v \in U$. □

**Lemma 2.3.** *Let $R$ be a 2-torsion free ring and let $U$ be a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. If $d : R \longrightarrow R$ is an additive mapping satisfying $d(u^2) = 2ud(u)$ for all $u \in U$, then*

(i) $[u,v]d([u,v]) = 0$, *for all $u,v \in U$.*
(ii) $(u^2v - 2uvu + vu^2)d(v) = 0$, *for all $u,v \in U$.*

**Proof.** (i) From Lemma 2.2 (i) and (v), we have

$$d(uv + vu) \;=\; 2\{ud(v) + vd(u)\} \text{ and } [u,v]\left(d(uv) - ud(v) - vd(u)\right) = 0$$

respectively. Combining these two results we find that

$$[u,v]\left(d(vu) - ud(v) - vd(u)\right) = 0\,, \quad \text{for all } u,v \in U\,. \tag{2.7}$$

Further, combining of (2.7) and Lemma 2.2 (v) yields that $[u,v]d([u,v]) = 0$.

(ii) For any $v,u \in U$, we have $d\left([u,v]^2\right) = 2[u,v]d\left([u,v]\right)$. Now application of Lemma 2.3 (i), gives that

$$d\left([u,v]^2\right) \;=\; 0\,, \quad \text{for all } u,v \in U\,. \tag{2.8}$$

Since $2uv \in U$, replacing $u$ by $2vu$ in $uv + vu \in U$ and $uv - vu \in U$ and adding the results so obtained we find that $4vuv \in U$ for all $u,v \in U$. Thus in view of Lemma 2.2 (i), we have

$$4d\left(u(uvu) + (vuv)u\right) \;=\; 8\{ud(vuv) + vuvd(u)\}$$

This implies that $d\,(u(vuv)+(vuv)u) = 2\{ud(vuv)+vuvd(u)\}$. Now application of (2.8) yields that

$$\begin{aligned} 0 &= d([u,v]^2) \\ &= d(u(vuv)+(vuv)u) - d(uv^2u) - d(vu^2v) \\ &= 2\{ud(vuv)+vuvd(u)\} - u^2d(v^2) - 3uv^2d(u) \\ &\quad + v^2ud(u) - v^2d(u^2) - 3vu^2d(v) + u^2vd(v) \\ &= -3(u^2v-2uvu+vu^2)d(v) - (uv^2-2vuv+v^2u)d(u) \end{aligned}$$

and hence,

$$(2.9)\quad (uv^2-2vuv+v^2u)d(u) + 3(u^2v-2uvu+vu^2)d(v) = 0\,,\quad \text{for all } u,v\in U\,.$$

In view of Lemma 2.2 (iv), we have

$$(2.10)\qquad (u^2v-2uvu+vu^2)d(u) = 0\,,\quad \text{for all } u,v\in U\,.$$

Replacing $u$ by $u+v$ in (2.10), we find that

$$\{(u^2v-2uvu+vu^2)-(v^2u-2vuv+uv^2)\}(d(u)+d(v)) = 0\,,\quad \text{for all } u,v\in U\,.$$

Now, using (2.10) in the above expression, we have

$$(2.11)\quad (u^2v-2uvu+vu^2)d(v) - (v^2u-2vuv+uv^2)d(u) = 0\,,\quad \text{for all } u,v\in U\,.$$

Combining (2.9) and (2.11), and using the fact that $R$ is 2-torsion free, we obtain $(u^2v-2uvu+vu^2)d(v)=0$. Thus in view of (2.11), we get the required result. □

## 3. Main Result

The main result of the present paper states as follows:

**Theorem.** *Let $R$ be a 2-torsion free prime ring and let $U$ be a Lie ideal of $R$ such that $u^2\in U$. If $d: R\longrightarrow R$ is an additive mapping such that $d(u^2)=2ud(u)$ for all $u\in U$, then $d(uv)=ud(v)+vd(u)$ for all $u,v\in U$.*

**Proof.** If $U$ is a commutative Lie ideal of $R$, then by using the same arguments as used in the proof of Lemma 1.3 of [8], $U\subset Z$. Hence using Lemma 2.2 (i), we find that $2d(uv)=2\{ud(v)+vd(u)\}$. But since char $R\neq 2$, we find that $d(uv)=ud(v)+vd(u)$ for all $u,v\in U$. Hence onward we shall assume that $U$ is a noncomutative Lie ideal of $R$ - i.e. $U\not\subset Z$. □

Now, by Lemma 2.2 (iv), we have

$$(3.1)\qquad (u^2v-2uvu+vu^2)d(u) = 0\,,\quad \text{for all } u,v\in U\,.$$

Replacing $u$ by $[u_1,w]$ in (3.1), we get

$$([u_1,w]^2v)d([u_1,w]) - 2([u_1,w]v[u_1,w])\;d([u_1,w]) + (v[u_1,w]^2d([u_1,w]) = 0\,,$$

for all $u,v,u_1,w\in U$.

Now, application of Lemma 2.3 (i), yields that $[u_1,w]^2Ud([u_1,w])=0$. Hence by Lemma 2.1 either $[u_1,w]^2=0$ or $d([u_1,w])=0$. If for some $u_1,w\in U$, $d([u_1,w])=0$ - i.e. $d(u_1w)=d(wu_1)$, then by using Lemma 2.2 (i) and the fact that char $R \neq 2$, we get $d(u_1 w) = u_1 d(w) + wd(u_1)$. On the other hand let $[u_1, w]^2 = 0$, for some $u_1, w \in U$. By Lemma 2.3 (ii), we get

$$(u^2 v - 2uvu + vu^2)d(v) = 0\,, \qquad \text{for all } u, v \in U\,. \tag{3.2}$$

Replacing $v$ by $[u_1, w]$ in (3.2), we get

$$(u^2[u_1, w])d([u_1, w]) - 2(u[u_1, w]u)d([u_1, w]) + ([u_1, w]u^2)d([u_1, w]) = 0\,,$$

for all $u \in U$.

Again apply Lemma 2.3 (i), to get

$$([u_1, w]u^2)d([u_1, w]) - 2(u[u_1, w]u)d([u_1, w]) = 0\,, \quad \text{for all } u \in U\,. \tag{3.3}$$

Linearizing (3.3) on $u$ and using (3.2), we have

$$\begin{aligned}([u_1, w]uv)d([u_1, w]) + ([u_1, w]vu)d([u_1, w]) - 2\{(u[u_1, w]v)\\ + (v[u_1, w]u)\}d([u_1, w]) = 0\,, \quad \text{for all } u, v \in U\,.\end{aligned} \tag{3.4}$$

Replace $u$ by $2uv_1$ in (3.4) and use the fact that $R$ is 2-torsion free, to get

$$\begin{aligned}([u_1, w]uv_1 v)\, d([u_1, w]) + ([u_1, w]vuv_1)\, d([u_1, w]) - 2\{(uv_1[u_1, w]v)\\ + (v[u_1, w]uv_1)\}d([u_1, w]) = 0\,, \quad \text{for all } u, v, v_1 \in U.\end{aligned}$$

Further, replacing $v_1$ by $[u_1, w]$ in the above expression and applying Lemma 2.3 (i) together with the fact that $[u_1, w]^2 = 0$, we find that $([u_1, w]u[u_1, w])\ v\ d([u_1, w]) = 0$ - i.e. $([u_1, w]u[u_1, w])Ud([u_1, w]) = 0$, for all $u \in U$. Thus by Lemma 2.1 either $d([u_1, w]) = 0$ or $[u_1, w]u[u_1, w] = 0$. If $d([u_1, w]) = 0$, then using the similar arguments as above we get the required result. On the other hand if $[u_1, w]u[u_1, w] = 0$ for all $u \in U$, then again by Lemma 2.1 we have $[u_1, w] = 0$. Further, application of Lemma 2.2 (i) yields that $2d(u_1 w) = 2\{u_1 d(w) + wd(u_1)\}$ and hence $d(u_1 w) = u_1 d(w) + wd(u_1)$. Hence in both the cases we find that $d(uv) = ud(v) + vd(u)$, for all $u, v \in U$. This completes the proof of the above theorem.

**Corollary.** *Let $R$ be a 2-torsion free prime ring and $d : R \longrightarrow R$ be a Jordan left derivation. Then $d$ is a left derivation.*

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