ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 207 – 212

THE NATURAL OPERATORS LIFTING VECTOR FIELDS TO GENERALIZED HIGHER ORDER TANGENT BUNDLES

WŁODZIMIERZ M. MIKULSKI

Abstract. For natural numbers r and n and a real number a we construct a natural vector bundle $T^{(r),a}$ over n-manifolds such that $T^{(r),0}$ is the (classical) vector tangent bundle $T^{(r)}$ of order r. For integers $r \geq 1$ and $n \geq 3$ and a real number $a < 0$ we classify all natural operators $T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$ lifting vector fields from n-manifolds to $T^{(r),a}$.

0. Let n and r be natural numbers and a be a real number. Consider the linear action $GL(n, \mathbf{R}) \times \mathbf{R} \to \mathbf{R}$ by $(B, x) \to |det(B)|^a x$. According to the theory of natural bundles, see e.g. [3], this action defines a natural vector bundle over n-manifolds. We will denote this natural bundle by $T^{(0,0),a}$. Given an n-manifold M let $T^{r*,a}M = \{j^r_x\sigma \mid \sigma \text{ is a local section of } T^{(0,0),a}M\ , \sigma(x) = 0, x \in M\}$ be the set of all r-jets of local sections of $T^{(0,0),a}M$ with target 0. It is a vector bundle over M with respect to the source projection. Let $T^{(r),a}M = (T^{r*,a}M)^*$ be the dual vector bundle. Every embedding $\varphi : M \to N$ of n-manifolds can be extended functorially to a vector bundle mapping $T^{r*,a}\varphi : T^{r*,a}M \to T^{r*,a}N$, $j^r_x\sigma \to j^r_{\varphi(x)}(T^{(0,0),a}\varphi \circ \sigma \circ \varphi^{-1})$, and (next) it can be extended to a vector bundle mapping $T^{(r),a}\varphi = ((T^{r*,a}\varphi)^*)^{-1} : T^{(r),a}M \to T^{(r),a}N$ over φ, and we obtain a natural vector bundle $T^{(r),a}$ over n-manifolds. $T^{(r),0}$ is the (classical) vector tangent bundle $T^{(r)}$ of order r over n-manifolds.

In this short note, we study the problem how a vector field X on an n-manifold M induces canonically a vector field $A(X)$ on $T^{(r),a}M$ for a natural number r and a real number $a < 0$. This problem is reflected in the concept of natural operators $A : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$ in the sense of Kolář, Michor and Slovák [3]. We prove the following theorem.

Theorem 1. *If $n \geq 3$ and $r \geq 1$ are integers and $a < 0$ is a negative real number, then the complete lifting $\mathcal{T}^{(r),a}$ of vector fields to $T^{(r),a}$ and the Liouville vector*

2000 *Mathematics Subject Classification*: 58A20, 53A55.
Key words and phrases: natural bundle, natural transformation, natural operator.
Received September 20, 1999.

field L on $T^{(r),a}$ form the basis (over $\mathbf{R}$) in the vector space of all natural operators $A : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$.

For $a = 0$ the classification is different. The main result of [4] says that if $n \geq 2$ and $r \geq 1$ are integers, then the vector space of all natural operators $A : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r)}$ is $(r+2)$-dimensional. (For $r = 1$ or $r = 2$ this fact was firstly proved in [5] or [1].) By the proof of Theorem 1 we reobtain the result of [4] for $n \geq 3$.

In this note the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \dots, x^n$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \dots, n$.

All manifolds and maps are assumed to be of class C^∞.

1. At first we study natural transformations $C : T^{(r),a} \to T^{(r),a}$ for $a \leq 0$ in the sense of [3].

Proposition 1. *If $n \geq 2$ and $r \geq 1$ are integers and $a \leq 0$ is a real number, then any natural transformation $C : T^{(r),a} \to T^{(r),a}$ over n-manifolds is proportional (by a real number) to the identity natural transformation.*

Proof. From now on the set of all $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $1 \leq |\alpha| \leq r$ will be denoted by $P(r, n)$.

Clearly, sections of $T^{(0,0),a}\mathbf{R}^n \tilde{=} \mathbf{R}^n \times \mathbf{R}$ are real valued functions on $\mathbf{R}^n$ satisfying respective transformation rules. Then any element from the fibre $T_0^{(r),a}\mathbf{R}^n$ of $T^{(r),a}\mathbf{R}^n$ over 0 is a linear combination of the $(j_0^r x^\alpha)^*$ for all $\alpha \in P(r, n)$, where the $(j_0^r x^\alpha)^*$ form the basis dual to the basis $j_0^r x^\alpha \in T_0^{r*,a}\mathbf{R}^n$.

Of course, any natural transformation C is (fully) determined by the contractions $\langle C(u), j_0^r x^\alpha \rangle \in \mathbf{R}$ for $u \in T_0^{(r),a}\mathbf{R}^n$ and $\alpha \in P(r, n)$, $j_0^r x^\alpha \in T_0^{r*,a}\mathbf{R}^n$.

If $\alpha = (\alpha_1, \dots, \alpha_n) \in P(r, n)$ with $\alpha_1 + \dots + \alpha_{n-1} \geq 1$ and $\tau \in \mathbf{R}$, then the diffeomorphism $\varphi_{\alpha,\tau} = (x^1, \dots, x^{n-1}, x^n - \tau(x^1)^{\alpha_1} \cdot \dots \cdot (x^{n-1})^{\alpha_{n-1}})$ sends $j_0^r((x^n)^{\alpha_n+1}) \in T_0^{r*,a}\mathbf{R}^n$ into $j_0^r((x^n + \tau(x^1)^{\alpha_1} \cdot \dots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1})$ (as $\varphi_{\alpha,\tau}^{-1} = (x^1, \dots, x^{n-1}, x^n + \tau(x^1)^{\alpha_1} \cdot \dots \cdot (x^{n-1})^{\alpha_{n-1}})$ and $det(Jac_0(\tau_{-\varphi_{\alpha,\tau}(y)} \circ \varphi_{\alpha,\tau} \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$, where $\tau_y : \mathbf{R}^n \to \mathbf{R}^n$ is the translation by y). Then by the naturality of C with respect to the diffeomorphisms $\varphi_{\alpha,\tau}$, the values $\langle C(u), j_0^r((x^n + \tau(x^1)^{\alpha_1} \cdot \dots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1}) \rangle$ for $u \in T_0^{(r),a}\mathbf{R}^n$ and $\tau \in \mathbf{R}$ are determined by the values $\langle C(u), j_0^r((x^n)^{\alpha_n+1}) \rangle$ for $u \in T_0^{(r),a}\mathbf{R}^n$. On the other hand, given $u \in T_0^{(r),a}\mathbf{R}^n$ the value $\frac{1}{\alpha_n+1}\langle C(u), j_0^r x^\alpha \rangle$ is the coefficient on τ of the polynomial $\langle C(u), j_0^r((x^n + \tau(x^1)^{\alpha_1} \cdot \dots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1}) \rangle$ with respect to τ. Therefore the values $\langle C(u), j_0^r x^\alpha \rangle$ for $u \in T_0^{(r),a}\mathbf{R}^n$ are determined by the values $\langle C(u), j_0^r((x^n)^{\alpha_n+1}) \rangle$ for $u \in T_0^{(r),a}\mathbf{R}^n$. Then C is fully determined by the values $\langle C(u), j_0^r((x^n)^i) \rangle$ for $u \in T_0^{(r),a}\mathbf{R}^n$ and $i = 1, \dots, r$.

For $i \in \{1, \dots, r\}$ the diffeomorphism $\varphi_i = (x^1 - (x^n)^i, x^2, \dots, x^n)$ sends $j_0^r(x^1) \in T_0^{r*,a}\mathbf{R}^n$ into $j_0^r(x^1 + (x^n)^i)$ (as $\varphi_i^{-1} = (x^1 + (x^n)^i, x^2, \dots, x^n)$ and $det(Jac_0(\tau_{-\varphi_i(y)} \circ \varphi_i \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$). Then by the naturality of C with respect to φ_i, the values $\langle C(u), j_0^r((x^n)^i) \rangle$ for $u \in T_0^{(r),a}\mathbf{R}^n$ are determined by the values $\langle C(u), j_0^r(x^1)\rangle$ for $u \in T_0^{(r),a}\mathbf{R}^n$. Then C is determined by the values $\langle C(u), j_0^r(x^1)\rangle \in \mathbf{R}$ for $u \in T_0^{(r),a}\mathbf{R}^n$, $j_0^r(x^1) \in T_0^{r*,a}\mathbf{R}^n$.

So, we will study the real valued function F given by $F((\mu_\alpha)_{\alpha\in P(r,n)}) := \langle C(\sum_\alpha \mu_\alpha \cdot (j_0^r x^\alpha)^*), j_0^r x^1\rangle$, $\mu_\alpha \in \mathbf{R}$, $\alpha \in P(r,n)$, $j_0^r(x^1) \in T_0^{r*,a}\mathbf{R}^n$

For any $t \in \mathbf{R}_+$ and any $\alpha \in P(r,n)$ the homothety $a_t = (tx^1, \dots, tx^n)$ sends $j_0^r x^\alpha \in T_0^{r*,a}\mathbf{R}^n$ into $t^{na-|\alpha|} j_0^r x^\alpha$, i.e. $(j_0^r x^\alpha)^*$ into $t^{|\alpha|-na} \cdot (j_0^r x^\alpha)^*$. Then by the naturality of C with respect to the homotheties a_t for $t \in \mathbf{R}_+$ we obtain the homogeneity condition $F(t^{|\alpha|-na}\mu_\alpha) = t^{1-na}F(\mu_\alpha)$. Then (since $na \le 0$) by the homogeneous function theorem, see [3], $F(\mu_\alpha)$ is the linear combination of the μ_α for $|\alpha| = 1$. Similarly, by the naturality of C with respect to the homotheties $b_t = (x^1, tx^2, \dots, tx^n)$ for $t \in \mathbf{R}_+$ we obtain $F(t^{\alpha_2+\dots+\alpha_n-(n-1)a}\mu_\alpha) = t^{-(n-1)a}F(\mu_\alpha)$. Then $F(\mu_\alpha)$ is proportional to $\mu_{(1,0,\dots,0)}$.

Hence the vector space of all natural transformations $C : T^{(r),a} \to T^{(r),a}$ over n-manifolds has dimension ≤ 1. This ends the proof of the proposition. $\square$

2. We are now in position to prove Theorem 1. Let $A : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$ be a natural operator, where $r \ge 1$ and $n \ge 1$ are integers and $a \le 0$. (We assume $a \le 0$ because we want to reobtain the result of [4].)

At first we prove that there exists a number $\lambda_A \in \mathbf{R}$ such that $A - \lambda_A T^{(r),a} : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$ is a vertical operator.

If $a = 0$, the G_n^{r+1}-space $S = T_0^{(r),a}\mathbf{R}^n$ corresponding to $T^{(r),a}$ is naturally contractible to $q = 0 \in S$ in the sense of Definition 1 in [2], and we can apply Proposition 1 in [2]. If $a < 0$, then the G_n^{r+1}-space $S = T_0^{(r),a}\mathbf{R}^n$ can not be naturally contractible, and we can not apply Proposition 1 in [2]. (For example, the curve $\gamma_{(j_0^r x^1)^*} : \mathbf{R} \to S$, $\gamma_{(j_0^r x^1)^*}(t) = T^{(r),a}(tid_{\mathbf{R}^n})((j_0^r x^1)^*) = t|t|^{-na} \cdot (j_0^r x^1)^*$ is not smooth at $t = 0$ for many $a < 0$, e.g. $-na = \frac{1}{2}$. Hence the property (ii) of Definition 1 in [2] is not satisfied.) In this case we modify the proof of Proposition 1 in [2] as follows. We define $h : \mathbf{R} \times S \to T_0\mathbf{R}^n = \mathbf{R}^n$ by $h(\lambda, u) = T\pi \circ A(\lambda\partial_1)(u)$, $\lambda \in \mathbf{R}$, $u \in S$, where $\pi : T^{(r),a}\mathbf{R}^n \to \mathbf{R}^n$ is the bundle projection. Since A is natural, h is equivariant with respect to the homotheties $a_t = tid_{\mathbf{R}^n}$, $t \in \mathbf{R}_+$. Then we obtain the homogeneity condition $h(t\lambda, \sum_\alpha t^{|\alpha|-na}\mu_\alpha \cdot (j_0^r x^\alpha)^*) = th(\lambda, \sum_\alpha \mu_\alpha \cdot (j_0^r x^\alpha)^*)$, $\mu_\alpha \in \mathbf{R}$, $\alpha \in P(r,n)$. Then, since $|\alpha| - na > 1$ for any $\alpha \in P(r,n)$, the homogeneous function theorem imply $h(\lambda, u) = h(\lambda, 0) = \lambda v$ for some $v \in \mathbf{R}^n$. Next, by the naturality of A with respect to the $b_t = (x^1, tx^2, \dots, tx^n)$ for $t \in \mathbf{R}_+$ (all b_t preserve ∂_1), we obtain that $h(1,u) = h(1,0) = \lambda_A \partial_{1|0}$ for some real number λ_A. Then $(A - \lambda_A T^{(r),a})(\partial_1)$ is vertical over 0. Hence $A - \lambda_A T^{(r),a}$ is a vertical operator.

Define a natural transformation $C_A := pr_2 \circ (A - \lambda_A T^{(r),a})(0) : T^{(r),a}M \to T^{(r),a}M$ for any n-manifold M, where 0 is the zero vector field on M and $pr_2 : VT^{(r),a}M \doteq T^{(r),a}M \times_M T^{(r),a}M \to T^{(r),a}M$ is the projection onto second factor. By Proposition 1, there exists $\mu_A \in \mathbf{R}$ such that $C_A = \mu_A id$.

Denote $B := A - \lambda_A T^{(r),a} - \mu_A L$. Then B is vertical and

$$B(0) = 0 \in \mathcal{X}(T^{(r),a}M) \text{ for any } n\text{-manifold } M\,. \tag{2.1}$$

It remains to prove that if $n \geq 3$ and $r \geq 1$ are integers and $a < 0$ (or $a = 0$), then the vector space of all natural operators $B : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$ of vertical type satisfying the condition (2.1) has dimension 0 (or $\leq r$).

Let $B : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$ be a vertical natural operator satisfying the condition (2.1). Assume $n \geq 3$, $r \geq 1$ and $a \leq 0$.

Define $\tilde{B} : \mathbf{R} \times T_0^{(r),a}\mathbf{R}^n \to T_0^{(r),a}\mathbf{R}^n$, $\tilde{B}(\lambda, u) = pr_2 \circ B(\lambda\partial_1)(u)$, $\lambda \in \mathbf{R}$, $u \in T_0^{(r),a}\mathbf{R}^n$, where pr_2 is as above. It is well-known that B is uniquely determined by $\tilde{B}(1, .) = pr_2 \circ B(\partial_1)_{|T_0^{(r),a}\mathbf{R}^n}$. So, we will study $\tilde{B}$.

For $\alpha \in P(r,n)$ we define $\tilde{B}_\alpha : \mathbf{R} \times T_0^{(r),a}\mathbf{R}^n \to \mathbf{R}$ by $\tilde{B} = \sum_{\alpha \in P(r,n)} \tilde{B}_\alpha \cdot (j_0^r x^\alpha)^*$. By the naturality of B with respect to the homotheties $a_t = tid_{\mathbf{R}^n}$ for $t \in \mathbf{R}_+$ we have the homogeneity condition $\tilde{B}_\alpha(t\lambda, \sum_\beta t^{|\beta|-na}\mu_\beta \cdot (j_0^r x^\beta)^*) = t^{|\alpha|-na}\tilde{B}_\alpha(\lambda, \sum_\beta \mu_\beta \cdot (j_0^r x^\beta)^*)$, $\mu_\beta \in \mathbf{R}$, $\beta \in P(r,n)$. By (2.1), $\tilde{B}_\alpha(0, .) = 0$ for any $\alpha \in P(r,n)$. Now, since $-na \geq 0$, from the homogeneous function theorem we deduce that $\tilde{B}_\alpha(\lambda, \sum_{\beta \in P(r,n)} \mu_\beta \cdot (j_0^r x^\beta)^*)$ is the linear combination of monomials in λ and the μ_β for $\beta \in P(r,n)$ with $|\beta| \leq |\alpha| - 1$. Hence for all $\mu_\beta \in \mathbf{R}$ we have

$$\tilde{B}(1, \sum_{\beta \in P(r,n)} \mu_\beta \cdot (j_0^r x^\beta)^*) = \tilde{B}(1, \sum_{\beta \in P(r-1,n)} \mu_\beta \cdot (j_0^r (x^\beta)^*)\,. \tag{2.2}$$

Now, we prove that $\tilde{B}(1, u) = \tilde{B}(1, 0)$ for all $u \in T_0^{(r),a}\mathbf{R}^n$.

Assume the contrary. Then by (2.2), $r \geq 2$. Let $k \geq 1$ be the minimal number such that there exists $\beta^o \in P(r,n)$ with $|\beta^o| = k$ such that $\Phi((\mu_\beta)_{\beta \in P(r,n)}) := \tilde{B}(1, \sum_{\beta \in P(r,n)} \mu_\beta \cdot (j_0^r x^\beta)^*)$ depends essentially on μ_{β^o}, i.e. $\frac{\partial}{\partial \mu_{\beta^o}}\Phi \neq 0$. (Then $r - k \geq 1$.) We fix $\beta^o = (\beta_1^o, \dots, \beta_n^o)$ as above. Let $j^o \in \{1, \dots, n\}$ be such that $\beta_{j^o}^o \geq 1$.

We produce a contradiction. Let $l^o \in \{1, \dots, n\} \setminus \{1, j^o\}$. (Such l^o exists as $n \geq 3$.) Let $\varphi = (x^1, \dots, x^{j^o} + (x^{l^o})^{r-k+1}, \dots, x^n)$ (only the j^o-position is exceptional). It is a diffeomorphism preserving both ∂_1 and $0 \in \mathbf{R}^n$. It is easily seen that $\varphi^{-1} = (x^1, \dots, x^{j^o} - (x^{l^o})^{r-k+1}, \dots, x^n)$ and that $det(Jac_0(\tau_{-\varphi^{-1}(y)} \circ \varphi^{-1} \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$, where τ_y is the translation by y. Denote $\tilde{\varphi} := T^{(r),a}\varphi$ and $\tilde{B}_1 = \tilde{B}(1, .)$. We say that $(j_0^r x^\beta)^*$, where $\beta \in P(r,n)$, is not essential if $|\beta| < k$ or $|\beta| = r$. It will be proved below that

$$\begin{aligned}
\Phi((\mu_\beta)_{\beta \in P(r,n)}) &= \tilde{B}_1(\sum_{\beta \in P(r-1,n), |\beta| \geq k} \mu_\beta \cdot (j_0^r x^\beta)^*) \\
&= \tilde{B}_1(\sum_{\beta \in P(r-1,n), |\beta| \geq k} \mu_\beta \cdot (j_0^r x^\beta)^* - \frac{\mu_{\beta^o}}{\beta_{j^o}^o}(j_0^r(x^{\beta^o - 1_{j^o}}(x^{l^o})^{r-k+1}))^*) \\
&= \tilde{\varphi}^{-1} \circ \tilde{B}_1(\tilde{\varphi}(\sum_{\beta \in P(r-1,n), |\beta| \geq k} \mu_\beta \cdot (j_0^r x^\beta)^* - \frac{\mu_{\beta^o}}{\beta_{j^o}^o}(j_0^r(x^{\beta^o - 1_{j^o}}(x^{l^o})^{r-k+1}))^*))
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\varphi}^{-1} \circ \tilde{B}_1\Big(\sum_{\beta \in P(r-1,n), |\beta| \geq k} \mu_\beta \cdot (j_0^r x^\beta)^* - \mu_{\beta^o} \cdot (j_0^r x^{\beta^o})^* + \dots \Big) \\
&= \tilde{\varphi}^{-1} \circ \tilde{B}_1\Big(\sum_{\beta \in P(r-1,n), |\beta| \geq k} \mu_\beta \cdot (j_0^r x^\beta)^* - \mu_{\beta^o} \cdot (j_0^r x^{\beta^o})^* \Big) ,
\end{aligned}
$$

where the dots is the linear combination of the not essential $(j_0^r x^\beta)^*$'s. Then Φ is independent of μ_{β^o}, i.e. we have a contradiction.

Let us explain the above equalities.

The first, the second and the last equalities are consequences of the formula (2.2), the definition of k, the definition of β^o, the definition of not essential $(j_0^r x^\beta)^*$'s and the equality $(|\beta^o| - 1) + (r - k + 1) = r$. The third equality is a consequence of the invariancy of B and ∂_1 with respect to φ. The fourth equality is a consequence of the following two facts: (a) For any $\beta \in P(r-1,n)$ with $|\beta| \geq k$ the diffeomorphism φ sends $(j_0^r x^\beta)^*$ into $(j_0^r x^\beta)^* + \dots$, where the dots denote the linear combination of the $(j_0^r x^\alpha)^*$ for $|\alpha| < k$. (b) The diffeomorphism φ sends $(j_0^r (x^{\beta^o - 1_{j^o}} (x^{l^o})^{r-k+1}))^*$ into $\beta^o_{j^o} \cdot (j_0^r x^{\beta^o})^* + \dots$, where the dots denote the linear combination of the $(j_0^r x^\alpha)^*$ for $|\alpha| < k$ or $|\alpha| = r$.

To prove the fact (a) we consider $\beta, \alpha \in P(r-1,n)$ with $|\alpha| \geq k$ and $|\beta| \geq k$. We have $\langle T^{(r),a}\varphi((j_0^r x^\beta)^*), j_0^r x^\alpha \rangle = \langle (j_0^r x^\beta)^*, j_0^r(T^{(0,0),a}(\varphi^{-1}) \circ x^\alpha \circ \varphi)\rangle = \langle (j_0^r x^\beta)^*, j_0^r(x^\alpha \circ \varphi)\rangle$ because of the Jacobian argument. But $j_0^r(x^\alpha \circ \varphi) = j_0^r((x^1)^{\alpha_1} \cdot \ldots \cdot (x^{j^o} + (x^{l^o})^{r-k+1})^{\alpha_{j^o}} \cdot \ldots \cdot (x^n)^{\alpha_n}) = j_0^r x^\alpha + \dots$, where the dots is the linear combination of the $j_0^r x^\gamma$ with $|\gamma| = r$. Hence $\langle T^{(r),a}\varphi((j_0^r x^\beta)^*), j_0^r x^\alpha \rangle = \delta^\beta_\alpha$ (the Kronecker delta). This ends the proof of the fact (a). The proof of the fact (b) is quite similar. (We propose to study the contractions $\langle T^{(r),a}\varphi((j_0^r(x^{\beta^o - 1_{j^o}}(x^{l^o})^{r-k+1}))^*)$, $j_0^r x^\alpha \rangle$ for $\alpha \in P(r-1,n)$ with $|\alpha| \geq k$.)

We have proved that $\tilde{B}(1,u) = \tilde{B}(1,0) = \sum_{\alpha \in P(r,n)} \nu_\alpha \cdot (j_0^r x^\alpha)^*$ for any $u \in T_0^{(r),a}\mathbf{R}^n$, where ν_α are the real numbers. Now, using the invariancy of B and ∂_1 with respect to the $b_t = (x^1, tx^2, \dots, tx^n)$ for $t \in \mathbf{R}_+$ we get the condition $\sum_{\alpha \in P(r,n)} \nu_\alpha \cdot (j_0^r x^\alpha)^* = \sum_{\alpha \in P(r,n)} t^{\alpha_2 + \dots + \alpha_n - (n-1)a} \nu_\alpha \cdot (j_0^r x^\alpha)^*$ for any $t \in \mathbf{R}_+$. Then for $a = 0$ we have $\tilde{B}(1,u) = \sum_{i=1}^r \lambda_i \cdot (j_0^r((x^1)^i))^*$ for any $u \in T_0^{(r),a}\mathbf{R}^n$, where λ_i are the real numbers, and for $a < 0$ we have $\tilde{B}(1,u) = 0$ for any $u \in T_0^{(r),a}\mathbf{R}^n$. Hence the vector space of all vertical natural operators $A : T_{|\mathcal{M}_n} \rightsquigarrow TT^{(r),a}$ satisfying condition (2.1) has dimension $\leq r$ if $a = 0$, and it has dimension 0 if $a < 0$.

The proof of Theorem 1 is complete. □

3. Remark. (a) Let X be a vector field on an n-manifold M. If $s = 1, \dots, r$, we have a vector field $A^{(s)}(X)$ on $T^{(r)}M$ given by $A^{(s)}(X)_u = (u, \tilde{A}^{(s)}(X)(x)) \in T_x^{(r)}M \times T_x^{(r)}M = V_u T^{(r)}M \subset T_u T^{(r)}M$, $u \in T_x^{(r)}M$, $x \in M$, where $\tilde{A}^{(s)}(X)(x) : J_x^r(M,\mathbf{R})_0 \to \mathbf{R}$ is a linear map given by $\tilde{A}^{(s)}(X)(x)(j_x^r\gamma) = X \circ \dots \circ X\gamma(x)$, s-times of X, $\gamma : M \to \mathbf{R}$, $\gamma(x) = 0$. Clarly, the natural operators $T^{(r)}, L, A^{(1)}, \dots, A^{(r)}$ are linearly independent. Then they form the basis of the vector space of all natural operators $T_{\mathcal{M}_n} \rightsquigarrow TT^{(r)}$ if $n \geq 3$. Thus we have reobtained the result of [4].

(b) Starting from the action $GL(n,\mathbf{R})\times\mathbf{R}\to\mathbf{R}$ given by $(B,x)\to sgn(det(B))$ $|det(B)|^a x$ instead of the one from Item 0 and using the same construction as in Item 0, we can construct new natural vector bundle $\tilde{T}^{(r),a}$ over n-manifolds. Clearly, Theorem 1 is true for $\tilde{T}^{(r),a}$ instead of $T^{(r),a}$. (We use the same proof with $\tilde{T}^{(r),a}$ instead of $T^{(r),a}$.) More, for $n \geq 3$ and $r \geq 1$, the vector space of natural operators $T_{\mathcal{M}_n} \rightsquigarrow T\tilde{T}^{(r),0}$ is also 2-dimensional, i.e. the complete lifting $\tilde{T}^{(r),0}$ and the Liouville vector field L form the basis in the vector space of all natural operators $T_{|\mathcal{M}_n} \rightsquigarrow T\tilde{T}^{(r),0}$. (To see this, in the last acapit of the proof of Theorem 1 we use additionally the invariancy of $\tilde{B}$ with respect to the $\psi_t = (x^1,\dots,x^{n-1},-tx^n)$ for $t \in \mathbf{R}_+$. Since $\tilde{T}^{(r),0}\psi_t$ sends $(j^r_0((x^1)^i))^*$ into $-(j^r_0((x^1)^i))^*$, then we deduce that $\lambda_1 = \cdots = \lambda_r = 0$.)

References

[1] Doupovec, M., *Natural operators transforming vector fields to the second order tangent bundle*, Cas. pest. mat. **115** (1990), 64–72.

[2] Kolář, I., *On the natural operators transforming vector fields to the r-th tensor power*, Suppl. Rendiconti Circolo Mat. Palermo, **32**(II) (1993), 15–20.

[3] Kolář, I., Michor, P. W., Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, Berlin 1993.

[4] Mikulski, W. M., *Some natural operations on vector fields*, Rendiconti Math. Roma **12**(VII) (1992), 783–803.

[5] Sekizava, M., *Natural transformations of vector fields on manifolds to vector fields on tangent bundles*, Tsukuba J. Math. **12** (1988), 115–128.

Institute of Mathematics, Jagiellonian University
Kraków, Reymonta 4, POLAND
E-mail: `mikulski@im.uj.edu.pl`