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SOME λ–SEQUENCE SPACES DEFINED BY A MODULUS

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ABSTRACT. The main object of this paper is to introduce and study some sequence spaces which arise from the notation of generalized de la Vallée–Poussin means and the concept of a modulus function.

1. Introduction

The notion of a modulus function was introduced by Nakano [7]. Ruckle [8] used the idea of a modulus function to construct some spaces of complex sequences. Maddox [5] investigated and discussed some properties of the three sequence spaces defined using a modulus function f, which generalized the well-known spaces w_0, w and w_∞ of strongly summable sequences which were discussed by Maddox [4].

Recently E.Savas [9] generalized the concept of strong almost convergence by using a modulus function f and examined some properties of the corresponding new sequence spaces.

Quite recently E.Savas [10] defined almost λ–statistical convergence by using the notion of (V, λ)–summability to generalize the concept of statistical covergence.

It is quite natural to the expect that the sets of sequences that are strongly almost summable to zero, strongly almost summable, and strongly almost bounded by the de la Vallée–Poussin method can be defined by combining the concepts of a modulus function, the (V, λ) method and strong almost convergence just as the spaces of sequences that are strongly almost summable to zero, strongly almost summable and strongly almost bounded with respect to the modulus f were defined by Savas [9].

The main object of this paper is to study some sequence spaces which arise from the notation of generalized de la Vallée–Poussin means and the concept of a modulus function.

Let ω be the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$, and l_∞, c and c_0 denote the Banach spaces of sequences that are bounded, convergent and convergent to zero, respectively, normed as usual by $\|x\|_\infty = \sup_k |x_k|$. We write ϕ for the set of

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all finite sequences. Furthermore, e and $e^{(n)}$ $(n = 1, 2, \cdots)$ are the sequences with $e_k = 1$ $(k = 1, 2, \cdots)$ and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$.

The concept of *paranorm* is closely related to linear metric spaces. It is a generalization of that of absolute value. Let X be a linear space. A function $p : X \to \mathbb{R}$ is called *paranorm*, if

(P.1) $p(0) \geq 0$
(P.2) $p(x) \geq 0$ for all $x \in X$
(P.3) $p(-x) = p(x)$ for all $x \in X$
(P.4) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ (*triangle inequality*)
(P.5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ (*continuity of multiplication by scalars*).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total*. It is well known that the metric of any linear metric space is given by some total paranorm (cf. [11, Theorem 10.4.2, p. 183]).

A complete linear metric space is said to be a *Fréchet space*. A Fréchet sequence space X is said to be an *FK space* if its metric is stronger than the metric of ω on X, that is convergence in the sequence space X implies coordinatewise convergence. (The letters F and K stand for *F*réchet and *K*oordinate, the German word for coordinate.) Some authors include local convexity in the definition of a Fréchet space and also of an FK space. We do not, but follow the definition used by Maddox and Wilansky. An FK space $X \supset \phi$ is said to have AK if $\sum_{k=0}^{n} x_k e^{(k)} \to x$ $(n \to \infty)$ for all sequences $x = (x_k)_{k=0}^{\infty} \in X$. (AK stands for *A*bschnitts*k*onvergenz (sectional convergence).)

Throughout, let $\Lambda = (\lambda_n)$ be a nondecreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized *de la Vallée-Poussin means* of a sequence x are defined as

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k \quad \text{where } I_n = [n - \lambda_n + 1, n] \text{ for } n = 1, 2, \cdots .$$

We write

$$[V, \lambda]_0 = \left\{ x \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\},$$

$$[V, \lambda] = \{ x \in \omega : x - le \in [V, \lambda]_0 \text{ for some } l \in \mathbb{C} \}$$

and

$$[V, \lambda]_\infty = \left\{ x \in \omega : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty \right\}$$

for the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée–Poussin method. In the special case where $\lambda_n = n$ for $n = 1, 2, \cdots$, the sets $[V, \lambda]_0$, $[V, \lambda]$ and $[V, \lambda]_\infty$ reduce to the sets w_0, w and w_∞ introduced and studied by Maddox [3].

Following Ruckle [8] and Maddox [5], a modulus function f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x)=0$ if and only if $x=0$,
(ii) $f(x+y)\le f(x)+f(y)$ for all $x,y\ge 0$,
(iii) f is increasing,
(iv) f is continuous from the right at zero.

Since $|f(x)-f(y)|\le f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$. Furthermore, we have $f(nx)\le nf(x)$ for all $n\in\mathbb{N}$ from condition (ii), and so $f(x)=f(nx\frac{1}{n})\le nf(x/n)$, hence

$$\frac{1}{n}f(x)\le f(x/n) \text{ for all } n\in\mathbb{N}\,. \tag{1.1}$$

A modulus may be bounded or unbounded. For example, $f(x)=x^p$ for $0<p\le 1$ is unbounded, but $f(x)=x/(1+x)$ is bounded.

Maddox [4] introduced and studied the sets

$$[\hat{c}]_0=\left\{x\in\omega:\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|x_{k+m}|=0 \text{ uniformly in } m\right\}$$

and

$$[\hat{c}]=\{x\in\omega:x-le\in[\hat{c}]_0 \text{ for some } l\in\mathbb{C}\}$$

of sequences that are *strongly almost convergent to zero* and *strongly almost convergent*. We combine the concepts of a modulus function, the strong (V,λ) method and strong almost convergence to define the sets

$$[\hat{V},\lambda,f]_0=\left\{x\in\omega:\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_{k+m}|)=0 \text{ uniformly in } m\right\},$$

$$[\hat{V},\lambda,f]=\left\{x\in\omega:x-le\in[\hat{V},\lambda,f]_0 \text{ for some } l\in\mathbb{C}\right\}$$

and

$$[\hat{V},\lambda,f]_\infty=\left\{x\in\omega:\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_{k+m}|)<\infty\right\}.$$

When $\lambda_n=n$ for $n=1,2,\cdots$, then the sets $[\hat{V},\lambda,f]$ and $[\hat{V},\lambda,f]_0$ reduce to the sets $[\hat{c}(f)]$ and $[\hat{c}_0(f)]$ defined by Savas [9]. If we put $f(x)=x$, then $[\hat{V},\lambda,f]_0=[\hat{V},\lambda]_0$ and $[\hat{V},\lambda,f]=[\hat{V},\lambda]$.

2. Some auxiliary results

In this section, we give a few lemmas needed in the proofs of the main results.

Lemma 1. *For any modulus f there exists* $\lim_{t\to\infty}f(t)/t$ (*cf.* [6, Proposition 1]).

Lemma 2. *Let f be any modulus with*

$$\lim_{t\to\infty}\frac{f(t)}{t}=\alpha>0\,. \tag{2.1}$$

Then there is a constant $\beta>0$ such that

$$f(t)\geq\beta t \text{ for all } t\geq 0\,. \tag{2.2}$$

Proof. We assume that (2.1) holds. Then there is $t_0>0$ such that

$$f(t)\geq\frac{\alpha}{2}t \text{ for all } t\geq t_0\,. \tag{2.3}$$

Now let $1\leq t\leq t_0$. Then by condition (iii) of a modulus

$$f(t)\geq f(1)=\frac{1}{t}f(1)t\geq\frac{f(1)}{t_0}t\,. \tag{2.4}$$

Finally, let $0<t<1$. Then there is $n\in\mathbb{N}$ such that $1/(n+1)<t\leq 1/n$, and by condition (iii) of a modulus and (1.1)

(2.5)
$$f(t)\geq f(\frac{1}{n+1})\geq\frac{1}{n+1}f(1)=\frac{n}{n+1}\frac{1}{n}f(1)\geq\frac{n}{n+1}f(1)t\geq\frac{1}{2}f(1)t\,.$$

We put $\beta=\min\{\alpha/2,f(1)/t_0,1/2f(1)\}>0$. Then (2.2) follows from (2.3), (2.4) and (2.5). □

Lemma 3. *Let f be any modulus and $0<\delta<1$. Then*

$$f(x)\leq 2f(1)\delta^{-1}x \text{ for all } x\geq\delta\,. \tag{2.6}$$

Proof. Inequality (2.6) follows by a straightforward computation using the properties of a modulus function. □

Lemma 4. *Let $x\in[\hat{V},\lambda,f]$. Then there is a unique complex number l with $x-le\in[\hat{V},\lambda,f]_0$.*

Proof. Let $x\in[\hat{V},\lambda,f]$ and $x-le,x-l'e\in[\hat{V},\lambda,f]_0$. Then given $\varepsilon>0$ there is $n\in\mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_k-l|)<\varepsilon/2 \quad\text{and}\quad \frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_k-l'|)<\varepsilon/2\,.$$

This implies

$$f(|l-l'|)\leq\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_k-l|)+\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_k-l'|)<\varepsilon\,.$$

Since $\varepsilon>0$ was arbitrary, this implies $l=l'$ by conditions (i) and (iv) of a modulus. □

Now we give some inclusions.

Lemma 5. *Let f be any modulus. Then*

(a) $$[\hat{V},\lambda,f]_\infty=l_\infty(f)=\{x\in\omega:(f(|x_k|))_{k=1}^\infty\in l_\infty\};$$

(b) $$[\hat{V},\lambda,f]_0\subset[\hat{V},\lambda,f]\subset[\hat{V},\lambda,f]_\infty.$$

Proof. (a) Let $x \in [\hat{V}, \lambda, f]_\infty$. Then there is a constant $M > 0$ such that

$$\frac{1}{\lambda_1} f(|x_{1+m}|) \le \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m}|) \le M \text{ for all } m\,,$$

and so $(f(|x_k|))_{k=1}^\infty \in l_\infty$.
Conversely, let $x \in l_\infty(f)$. Then there is a constant $M > 0$ such that $f(|x_j|) \le M$ for all j, and so

$$\frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m}|) \le M \frac{1}{\lambda_n} \sum_{k \in I_n} 1 \le M \text{ for all } m \text{ and } n\,.$$

Thus $x \in [\hat{V}, \lambda, f]_\infty$.

(b) Obviously $[\hat{V}, \lambda, f]_0 \subset [\hat{V}, \lambda, f]$.
Let $x \in [\hat{V}, \lambda, f]$. Then there are a complex number l and a positive integer n_0 such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m} - l|) \le 1 \text{ for all } n \ge n_0 \text{ and for all } m \in \mathbb{N}\,,$$

and so

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m}|) &\le \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m} - l|) + \frac{1}{\lambda_n} \sum_{k \in I_n} f(|l|) \\ &\le 1 + f(|l|) \text{ for all } n \ge n_0 \text{ and for all } m \in \mathbb{N}\,. \end{aligned}$$

Furthermore

$$\frac{1}{\lambda_{n_0}} f(|x_{n_0+m}|) \le \frac{1}{\lambda_{n_0}} \sum_{k \in I_{n_0}} f(|x_{k+m}|) \le 1 + f(|l|) \text{ for all } m \in \mathbb{N}\,,$$

hence $f(|x_{n_0+m}|) \le \lambda_{n_0}(1 + f(|l|))$ for all m. Thus $x \in l_\infty(f)$. □

3. The main results

In this section we prove the main results.

Theorem 1. *The spaces $[\hat{V}, \lambda, f]_0$ and $[\hat{V}, \lambda, f]$ are paranormed FK spaces with the paranorm p defined by*

$$p(x) = \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m}|)\,, \tag{3.1}$$

$[\hat{V}, \lambda, f]_0$ has AK, and every sequence $x = (x_k)_{k=1}^\infty \in [\hat{V}, \lambda, f]$ has a unique representation

$$x = le + \sum_{k=1}^\infty (x_k - l) e^{(k)} \tag{3.2}$$

$$\text{where } l \in \mathbb{C} \text{ is such that } x - le \in [\hat{V}, \lambda, f]_0\,.$$

Proof. First we show that $[\hat{V},\lambda,f]$ is a paranormed FK space with p defined in (3.1). The proof for $[\hat{V},\lambda,f]_0$ is exactly the same.
By Lemma 5 (b), p is defined on $[\hat{V},\lambda,f]$. Obviously $p(x)\geq 0$ for all $x\in[\hat{V},\lambda,f]$. Furthermore, $p(x)=0$ implies $f(|x_j|)=0$ for all j, and so $x=0$ by condition (i) of a modulus. Obviously $p(-x)=p(x)$ for all $x\in[\hat{V},\lambda,f]$. Let $x,y\in[\hat{V},\lambda,f]$. Then by conditions (iii) and (ii) of a modulus

$$\begin{aligned}p(x+y)&\leq\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}\left(f(|x_{k+m}|)+f(|y_{k+m}|)\right)\\&\leq\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_{k+m}|)+\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}f(|y_{k+m}|)\leq p(x)+p(y)\,.\end{aligned}$$

Now let $p(x^{(r)})\to 0$ and $\mu_r\to\mu$ $(r\to\infty)$. Then the sequence (μ_r) is bounded, $|\mu_r|\leq M\in\mathbb{N}$ for all r, say. By conditions (iii) and (i) of a modulus

$$\begin{aligned}p(\mu_r x^{(r)})&=\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}f(|\mu_r x^{(r)}_{k+m}|)\leq\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}f(M|x^{(r)}_{k+m}|)\\&\leq M\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x^{(r)}_{k+m}|)=M\cdot p(x^{(r)})\to 0\quad(r\to\infty)\,.\end{aligned}$$

Finally, let $x\in[\hat{V},\lambda,f]$ be given and $\mu_r\to 0$ $(r\to\infty)$. Given $\varepsilon>0$, there is $n_0\in\mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_{k+m}-l|)<\varepsilon/2\text{ for all }n>n_0\text{ and for all }m\,. \tag{3.3}$$

Since $x\in l_\infty(f)$ by Lemma 5 (a) and since f is continuous, we can choose $r_0\in\mathbb{N}$ such that for all $r\geq r_0$

$$\frac{1}{\lambda_n}\sum_{k\in I_n}f(|\mu_r x_{k+m}|)<\varepsilon \tag{3.4}$$
$$\text{for all }n\text{ with }1\leq n\leq n_0\text{ and for all }m\,.$$

Now let $n>n_0$. By the continuity of f and since $\mu_r\to 0$ $(r\to\infty)$, there is $r_1\in\mathbb{N}$ such that

$$f(|\mu_r l|)<\varepsilon/2\quad\text{and}\quad|\mu_r|<1\quad\text{for all }r\geq r_1\,. \tag{3.5}$$

Then, by (3.3) and (3.5), for all $r\geq r_1$

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}f(|\mu_r x_{k+m}|)&\leq\frac{1}{\lambda_n}\sum_{k\in I_n}f(|\mu_r l|)+\frac{1}{\lambda_n}\sum_{k\in I_n}f(|\mu_r(x_{k+m}-l)|)\leq\\&\leq f(|\mu_r l|)+([|\mu_r|]+1)\frac{1}{\lambda_n}\sum_{k\in I_n}f(|x_{k+m}-l|)<\varepsilon/2+\varepsilon/2=\varepsilon\text{ for all }m\,.\end{aligned} \tag{3.6}$$

We choose $r_2=\max\{r_1,r_0\}$. Then, by (3.4) and (3.6), $p(\mu_r x)\leq\varepsilon$ for all $r\geq r_2$, that is $p(\mu_r x)\to 0$ for $r\to\infty$. Thus we have shown that p is a total paranorm.

Now we show that $[\hat{V},\lambda,f]$ is complete. Let $(x^{(r)})_{r=0}^{\infty}$ be a Cuachy sequence in $[\hat{V},\lambda,f]$. Then given $\varepsilon>0$ there is $r_0\in\mathbb{N}_0$ such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_{k+m}^{(r)}-x_{k+m}^{(s)}|)<\varepsilon \quad \text{for all } r,s\geq r_0 \text{ and for all } m \text{ and } n\,. \tag{3.7}$$

This implies

$$f(|x_j^{(r)}-x_j^{(s)}|)<\varepsilon \text{ for all } r,s\geq r_0 \text{ and for each } j\,. \tag{3.8}$$

Consequently, for each fixed j, the sequence $(x_j^{(r)})_{r=0}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$, hence convergent, $x_j=\lim_{r\to\infty}x_j^{(r)}$, say. For if $(x_{j_0}^{(r)})_{r=0}^{\infty}$ were not a Cauchy sequence for some j_0 then there would be subsequences $(x_{j_0}^{(r_k)})_{k=1}^{\infty}$ and $(x_{j_0}^{(s_k)})_{k=1}^{\infty}$ such that for some $c>0$ we have $|x_{j_0}^{(r_k)}-x_{j_0}^{(s_k)}|\geq c$ for all k, and so $f(|x_{j_0}^{(r_k)}-x_{j_0}^{(s_k)}|)\geq f(c)>0$ for all k, a contradiction to (3.8).
Fixing $r\geq r_0$ and letting s tend to infinity, we obtain from (3.7) by the continuity of f that

$$\frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_{k+m}^{(r)}-x_{k+m}|)\leq\varepsilon \text{ for all } r\geq r_0 \text{ and for all } m \text{ and } n\,, \tag{3.9}$$

that is

$$p(x^{(r)}-x)\leq\varepsilon \text{ for all } r\geq r_0\,. \tag{3.10}$$

Furthermore, for each r there exists $n_r\in\mathbb{N}$ such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_{k+m}^{(r)}-l^{(r)}|)<\varepsilon \text{ for all } n\geq n_r \text{ and for all } m\,. \tag{3.11}$$

Now let $r,s\geq r_0$ and $n_0=\max\{n_r,n_s\}$. Then by (3.7) and (3.11),

$$\begin{aligned}
f(|l^{(r)}-l^{(s)}|) &= \frac{1}{\lambda_{n_0}}\sum_{k\in I_{n_0}} f(|l^{(r)}-l^{(s)}|)\\
&\leq \frac{1}{\lambda_{n_0}}\sum_{k\in I_{n_0}} f(|x_{k+m}^{(r)}-l^{(r)}|)+\frac{1}{\lambda_{n_0}}\sum_{k\in I_{n_0}} f(|x_{k+m}^{(s)}-l^{(s)}|)\\
&\quad+\frac{1}{\lambda_{n_0}}\sum_{k\in I_{n_0}} f(|x_{k+m}^{(r)}-x_{k+m}^{(s)}|)\\
&<\varepsilon+\varepsilon+\varepsilon \text{ for all } m\,,
\end{aligned}$$

and again this implies $l^{(r)}\to l$ $(r\to\infty)$, say. From this, (3.10) and (3.11), we obtain

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_{k+m}-l|) &\leq \frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_{k+m}-x_{k+m}^{(r_0)}|)+\frac{1}{\lambda_n}\sum_{k\in I_n} f(|x_{k+m}^{(r_0)}-l^{(r_0)}|)\\
&\quad+\frac{1}{\lambda_n}\sum_{k\in I_n} f(|l-l^{(r_0)}|)<4\varepsilon \text{ for all sufficiently large } n \text{ and for all } m\,,
\end{aligned}$$

hence $x \in [\hat{V}, \lambda, f]$. This shows that $[\hat{V}, \lambda, f]$ is complete.
Now we show that $p(x^{(r)} - x) \to 0$ $(r \to \infty)$ implies $x_j^{(r)} \to x_j$ $(r \to \infty)$ for each j. Let $p(x^{(r)} - x) \to \infty$ $(r \to \infty)$. Then, given $\varepsilon > 0$, there is $r_0 \in \mathbb{N}_0$ such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m}^{(r)} - x_{k+m}|) < \varepsilon \text{ for all } r \geq r_0 \text{ and for all } m \text{ and } n\,.$$

This implies $f(|x_j^{(r)} - x_j|) < \varepsilon$ for all $r \geq r_0$ and for each j, and again, as above, this implies $x_j^{(r)} \to x_j$ $(r \to \infty)$ for each j.
This completes the proof that $[\hat{V}, \lambda, f]$ is a paranormed FK space.
To show that $[\hat{V}, \lambda, f]_0$ has AK, let $x \in [\hat{V}, \lambda, f]_0$ and $\varepsilon > 0$ be given. Then we can choose $n_0 \in \mathbb{N}_0$ such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m}|) < \varepsilon \text{ for all } n \geq n_0 \text{ and for all } m\,.$$

Now we choose $r_1 \in \mathbb{N}$ such that $r_1 - \lambda_{r_1} + 1 \geq n_0$. Let $r \geq r_1$ and $x^{[r]} = \sum_{k=1}^{r} x_k e^{(k)}$. Then

$$p(x^{[r]} - x) \leq \sup_{n \geq r_0,\, m} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m}|) \leq \varepsilon\,.$$

Finally, let $x = (x_k)_{k=1}^{\infty} \in [\hat{V}, \lambda, f]$ be given. By Lemma 4, there is a unique complex number l such that $y = x - le \in [\hat{V}, \lambda, f]_0$. Since $[\hat{V}, \lambda, f]_0$ has AK $y = \sum_{k=1}^{\infty} y_k e^{(k)} = \sum_{k=1}^{\infty} (x_k - l) e^{(k)}$ and so $x = le + \sum_{k=1}^{\infty} (x_k - l) e^{(k)}$. □

Theorem 2. (a) *For any modulus f, $[\hat{V}, \lambda] \subset [\hat{V}, \lambda, f]$ and $[\hat{V}, \lambda]_0 \subset [\hat{V}, \lambda, f]_0$.*
(b) *If $\lim_{t \to \infty} f(t)/t = \alpha > 0$ then $[\hat{V}, \lambda, f] = [\hat{V}, \lambda]$ and $[\hat{V}, \lambda, f]_0 = [\hat{V}, \lambda]_0$.*

Proof. (a) Let f be a modulus and $x \in [\hat{V}, \lambda]$. Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - l| < \varepsilon \text{ for all } n \geq n_0 \text{ and for all } m\,.$$

Since f is continuous, we may choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for all $t \in [0, \delta]$. Then, by Lemma 3,

$$\begin{aligned}
\frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_{k+m} - l|) &= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |x_{k+m} - l| \leq \delta}} f(|x_{k+m} - l|) + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |x_{k+m} - l| > \delta}} f(|x_{k+m} - l|) \\
&< \varepsilon + 2f(1)\delta^{-1} \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - l| \\
&< \varepsilon(1 + 2f(1)\delta^{-1}) \text{ for all } n \geq n_0 \text{ and for all } m\,.
\end{aligned}$$

Thus $x \in [\hat{V}, \lambda, f]$.
The inclusion $[\hat{V}, \lambda]_0 \subset [\hat{V}, \lambda, f]_0$ is shown in exactly the same way.

(b) Let $x \in [\hat{V}, \lambda, f]$. By Lemma 1, the limit $\lim_{t\to\infty} f(t)/t$ exists, and by Lemma 2, there is $\beta > 0$ such that $f(t) \geq \beta t$ for all $t \geq 0$, and so

$$\frac{1}{\lambda_n} \sum_{k\in I_n} |x_{k+m} - l| \leq \frac{1}{\beta\lambda_n} \sum_{k\in I_n} f(|x_{k+m} - l|) \text{ for all } n \text{ and for all } m\,.$$

Thus $x \in [\hat{V}, \lambda]$.
The part concerning the sets $[\hat{V}, \lambda, f]_0$ and $[\hat{V}, \lambda]_0$ is shown in exactly the same way. □

The idea of *statistical convergence* was introduced by Fast [2], and has been studied in number theory [1] and trigonometric series [13]. A sequence x is said to be *statistically convergent to a number* $l \in \mathbb{C}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0\,,$$

where for each set M, $|M|$ denotes the cardinality of M. In this case we write $s - \lim x = l$ or $x_k \to l(s)$, and s denotes the set of all statistically convergent sequences. In [10] the definition of statistical convergence was extended to the concept of almost λ–statistical convergence. A sequence x is said to be *almost λ–statistically convergent to the number l* if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_{k+m} - l| \geq \varepsilon\}| = 0 \text{ uniformly in } m\,.$$

In this case we write $\hat{s}_\lambda - \lim x = l$ or $x_k \to l(\hat{s}_\lambda)$ and $\hat{s}_\lambda = \{x \in \omega : \hat{s}_\lambda - \lim x = l \text{ for some } l \in \mathbb{C}\}$. If $\lambda_n = n$, this definition reduces to the concept of almost statistical convergence which was defined in Savas [10].

We establish a relation between the sets $\hat{s}_\lambda$ and $[\hat{V}, \lambda, f]$.

Theorem 3. *The inclusion $\hat{s}_\lambda \subset [\hat{V}, \lambda, f]$ holds if and only if f is bounded.*

Proof. We assume that f is bounded and $x \in \hat{s}_\lambda$. Then there is a constant M such that $f(x) \leq M$ for all $x \geq 0$. Let $\varepsilon > 0$ be given. We choose $\eta, \delta > 0$ such that $M\delta + f(\eta) < \varepsilon$. Since $x \in \hat{s}_\lambda$, there are $l \in \mathbb{C}$ and $n_0 = n_0(\eta, \delta) \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} |\{k \in I_n : |x_{k+m} - l| \geq \eta\}| < \delta \text{ for all } n \geq n_0 \text{ and for all } m\,.$$

Therefore

$$\begin{aligned}
\frac{1}{\lambda_n} \sum_{k\in I_n} f(|x_{k+m} - l|) &= \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |x_{k+m}-l|\geq\eta}} f(|x_{k+m} - l|) + \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |x_{k+m}-l|<\eta}} f(|x_{k+m} - l|) \\
&\leq M \frac{1}{\lambda_n} |\{k \in I_n : |x_{k+m} - l| \geq \eta\}| + f(\eta) \\
&< M\delta + f(\eta) < \varepsilon \text{ for all } n \geq n_0 \text{ and for all } m\,,
\end{aligned}$$

hence $x \in [\hat{V}, \lambda, f]$.
Conversely we assume that f is unbounded. Then there exists a positive sequence

$(v_k)_{k=1}^{\infty}$ with $f(v_k) = k^2$ for $k = 1, 2, \cdots$. We define the sequence x by $x_i = v_k$ if $i = k^2$ and $x_i = 0$ otherwise for $(i = 1, 2, \cdots)$. Then

$$\frac{1}{\lambda_n} \left|\{k \in I_n : |x_{k+m}| \geq \varepsilon\}\right| \leq \frac{1}{\lambda_n} \sqrt{\lambda_{n-1}} \text{ for all } n \text{ and } m\,,$$

and so $x \in \hat{s}_\lambda$. But obviously $x \notin [\hat{V}, \lambda, f]$. □

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