ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 171 – 181

CHARACTERIZATION OF POSETS OF INTERVALS

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Abstract. If $\mathcal{A}$ is a class of partially ordered sets, let $\mathcal{P}(\mathcal{A})$ denote the system of all posets which are isomorphic to the system of all intervals of $\mathbb{A}$ for some $\mathbb{A} \in \mathcal{A}$. We give an algebraic characterization of elements of $\mathcal{P}(\mathcal{A})$ for $\mathcal{A}$ being the class of all bounded posets and the class of all posets $\mathbb{A}$ satisfying the condition that for each $a \in \mathbb{A}$ there exist a minimal element u and a maximal element v with $u \leq a \leq v$, respectively.

For a partially ordered set $\mathbb{A}$ let Int $\mathbb{A}$ be the system of all intervals of $\mathbb{A}$; further, we put $\mathrm{Int}_0\ \mathbb{A} = \mathrm{Int}\ \mathbb{A} \cup \{\emptyset\}$. The systems Int $\mathbb{A}$ and $\mathrm{Int}_0\ \mathbb{A}$ are partially ordered by the set-theoretical inclusion.

These systems, particularly in the case when $\mathbb{A}$ is a lattice, have been investigated in several papers (cf. [1]-[12]). In [5], the algebraic characterization of $\mathrm{Int}_0\ \mathbb{L}$ for $\mathbb{L}$ being a lattice with a least or with a greatest element was given.

For each class $\mathcal{A}$ of partially ordered sets we denote by $\mathcal{P}(\mathcal{A})$ the class of all partially ordered sets $\mathbb{P}$ having the property that there exists $\mathbb{A}_{\mathbb{P}} \in \mathcal{A}$ such that Int $\mathbb{A}_{\mathbb{P}}$ is isomorphic to $\mathbb{P}$.

Let us denote by

$\mathcal{A}_\alpha$ - the class of all partially ordered sets $\mathbb{A}$ which have the least element and the greatest element;

$\mathcal{A}_\beta$ - the class of all partially ordered sets $\mathbb{A}$ such that for each $a \in \mathbb{A}$ there exists a minimal element u of $\mathbb{A}$ and a maximal element v of $\mathbb{A}$ with $u \leq a \leq v$.

In the present paper we give an algebraic characterization of elements of $\mathcal{P}(\mathcal{A}_\alpha)$ or $\mathcal{P}(\mathcal{A}_\beta)$, respectively.

The question of characterizing $\mathcal{P}(\mathcal{A}_t)$, where $\mathcal{A}_t$ is the class of all partially ordered sets, remains open.

2000 *Mathematics Subject Classification*: 06A06.
Key words and phrases: partially ordered set, interval.
The author was supported by the Slovak VEGA Grant No. 1/4379/97.
Received April 6, 1999.

1. The class $\mathcal{P}(\mathcal{A}_\alpha)$

We will deal with partially ordered sets $\mathbb{P}$ and $\mathbb{A}$ which have the underlying set P or A, respectively. The corresponding partial orders are denoted by $\leq$ or $\preceq$, respectively.

We recall that by an interval of a partially ordered set $\mathbb{P} = (P, \leq)$ a set $< a, b >= \{x \in P : a \leq x \leq b\}$ with $a, b \in P$, $a \leq b$ is meant. If $a = b$, we use the notation $<a>$ instead of $<a, a>$. The symbol $(a>$ will be used for the set $\{x \in P : x \leq a\}$. (Remark that $(a>$ need not be an interval.)

The system of all minimal and maximal elements of $\mathbb{P}$ will be denoted by Min $\mathbb{P}$ and Max $\mathbb{P}$, respectively.

An analogous terminology is applied for $\mathbb{A} = (A, \preceq)$.

Consider the following condition concerning the partially ordered set $\mathbb{P} = (P, \leq)$:

(β_1) if $x \in P$, then there exists $u \in$ Min $\mathbb{P}$ with $u \leq x$.

Theorem 1.1. *Let $\mathbb{P} = (P, \leq)$ be a partially ordered set. Then $\mathbb{P} \in \mathcal{P}(\mathcal{A}_\alpha)$ if and only if $\mathbb{P}$ has a greatest element I, it fulfils the condition (β_1) and there exist $o, i \in$ Min $\mathbb{P}$ and a dual isomorphism*

$$\varphi : <o, I> \to <i, I>$$

satisfying:

(1) *if $x \in P$, then $y = \sup\,\{x, o\}$, $z = \sup\,\{x, i\}$ exist and $x = \inf\,\{y, z\}$ holds;*
(2) *if $y \in < o, I >$, $z \in < i, I >$ and the set $\{y, z\}$ has a lower bound, then $\varphi^{-1}(z) \leq y$;*
(3) *if $y_1, y \in <o, I>$, $y_1 \leq y$, then $\inf\,\{y, \varphi(y_1)\}$ exists.*

Proof. Let $\mathbb{P} \in \mathcal{P}(\mathcal{A}_\alpha)$. Then $\mathbb{P}$ is isomorphic to Int $\mathbb{A}$ for a partially ordered set $\mathbb{A} = (A, \preceq)$ with a least element 0 and a greatest element 1. Evidently $\prec 0, 1 \succ$ is the greatest element of Int $\mathbb{A}$, Int $\mathbb{A}$ fulfils (β_1) and if we take $o = \prec 0 \succ$, $i = \prec 1 \succ$ and define φ by $\varphi(\prec 0, a \succ) = \prec a, 1 \succ$, φ is a dual isomorphism and the conditions (1)–(3) are satisfied. Hence $\mathbb{P}$ has the required properties, too.

The proof of the converse is made in several steps. So let $\mathbb{P}$ fulfil (β_1) and let I be the greatest element of $\mathbb{P}$, $o, i \in$ Min $\mathbb{P}$, φ be a dual isomorphism $<o, I> \to <i, I>$ such that (1)–(3) are satisfied. For the sake of brevity we will write $u \wedge v$ and $u \vee v$ instead of $\inf\{u, v\}$ and $\sup\{u, v\}$, respectively.

A. If $y \in < o, I >$, then $p = y \wedge \varphi(y)$ exists and $p \in$ Min $\mathbb{P}$. Moreover, $y = p \vee o$, $\varphi(y) = p \vee i$.

Let $y \in <o, I>$. Then $y \wedge \varphi(y)$ exists by (3). Let $y \wedge \varphi(y) = x$. By (β_1) there exists $p \in$ Min $\mathbb{P}$ with $p \leq x$. We have $p = y' \wedge z'$, where $y' = p \vee o$, $z' = p \vee i$ by (1). Obviously $y' \leq y$, $z' \leq \varphi(y)$. Now p is a lower bound of both $\{y', \varphi(y)\}$ and

$\{y, z'\}$, so that we have $y = \varphi^{-1}(\varphi(y)) \leq y'$, $\varphi^{-1}(z') \leq y$ by (2). The latter gives $z' \geq \varphi(y)$. We have $y = y'$, $z' = \varphi(y), x = p$.

B. Let $x \in P$, $x = u \wedge v$, $u \in <o, I>$, $v \in <i, I>$. Then $u = x \vee o$, $v = x \vee i$. Moreover, $x \in \text{Min } \mathbb{P}$ if and only if $v = \varphi(u)$.

Denote $y = x \vee o$, $z = x \vee i$ (the existence follows from (1)). Obviously $y \leq u$ and $z \leq v$. As $\{y, v\}$ has a lower bound, we have $\varphi^{-1}(v) \leq y$. Using A we obtain $\varphi^{-1}(v) \wedge v = r \in \text{Min } \mathbb{P}$. Since $\varphi^{-1}(v) \leq u$, we have $r \leq u \wedge v = x$. Hence $\{\varphi^{-1}(v), z\}$ has a lower bound and consequently $\varphi^{-1}(z) \leq \varphi^{-1}(v)$, which implies $z \geq v$. As also $z \leq v$ holds, we have $v = z = x \vee i$. The proof of $u = x \vee o$ is analogous.

If $v = \varphi(u)$, then $x \in \text{Min } \mathbb{P}$ by A. Conversely, let $x = u \wedge v \in \text{Min } \mathbb{P}$. Then evidently $\varphi^{-1}(v) \wedge v = x$ and also $u \wedge \varphi(u) = x$. So the set $\{\varphi^{-1}(v), \varphi(u)\}$ has a lower bound and consequently $u = \varphi^{-1}(\varphi(u)) \leq \varphi^{-1}(v)$, which implies $\varphi(u) \geq v$. But we have also $\varphi(u) \leq v$ and therefore $v = \varphi(u)$.

C. Let $x \in P$, $y = x \vee o$, $z = x \vee i$, $q = y \wedge \varphi(y)$, $p = \varphi^{-1}(z) \wedge z$. Then $x = p \vee q$.

As $x = y \wedge z$, we have $\varphi^{-1}(z) \leq y$ by (2), which implies $z \geq \varphi(y)$. Consequently $p, q \leq x$. Now let $x' \geq p, q$, $x' \in P$. In view of A we have $y = q \vee o \leq x' \vee o$, $z = p \vee i \leq x' \vee i$. Using (1) we obtain $x' = (x' \vee o) \wedge (x' \vee i) \geq y \wedge z = x$.

Let us remark that $I = o \vee i$ by C.

If $p \in \text{Min } \mathbb{P}$, set $\psi(p) = p \vee o$.

D. The mapping $\psi : \text{ Min } \mathbb{P} \to <o, I>$ is a bijection.

In view of A, ψ is onto. Let $p, q \in \text{Min } \mathbb{P}, p \vee o = q \vee o = y$. By (1), $p = y \wedge (p \vee i)$, $q = y \wedge (q \vee i)$ and using B we obtain $p \vee i = \varphi(y) = q \vee i$. Thus $p = q$.

Let $A = \text{Min } \mathbb{P}$ and define a partial order $\preceq$ in A by

$$p \preceq q \ (p, q \in \text{Min } \mathbb{P}) \Longleftrightarrow p \vee o \leq q \vee o.$$

Notice that ψ is now an isomorphism of $\mathbb{A} = (A, \preceq)$ onto $(<o, I>, \leq)$. The aim is to show that $\mathbb{P}$ is isomorphic to Int $\mathbb{A}$.

E. If $p, q \in A$, $p \preceq q$, then $p \vee q$ exists (in $\mathbb{P}$) and $p \vee q = (q \vee o) \wedge \varphi(p \vee o)$ holds.

The relation $p \vee o \leq q \vee o$ implies that $x = (q \vee o) \wedge \varphi(p \vee o)$ exists. By B we have $q \vee o = x \vee o$, $\varphi(p \vee o) = x \vee i$ and consequently $(x \vee o) \wedge \varphi(x \vee o) = (q \vee o) \wedge \varphi(q \vee o) = q$, $\varphi^{-1}(x \vee i) \wedge (x \vee i) = (p \vee o) \wedge \varphi(p \vee o) = p$. Using C we obtain $x = p \vee q$.

Now let us define $\Phi :$ Int $(A, \preceq) \to P$ by

$$\Phi(\prec p, q \succ) = p \vee q (= (q \vee o) \wedge \varphi(p \vee o)) \quad (p, q \in A, p \preceq q).$$

F. The mapping Φ is an isomorphism of (Int $(A, \preceq), \subseteq$) onto $\mathbb{P} = (P, \leq)$.

To prove that Φ is onto, let $x \in P$. Take p, q as in C. Then $p \vee o = \varphi^{-1}(x \vee i) \leq x \vee o = q \vee o$ by A, hence $p \preceq q$ and $\varphi(\prec p, q \succ) = p \vee q = x$ by C.

Further let $p \preceq q, p_1 \preceq q_1$. We will show that $\prec p, q \succ \subseteq \prec p_1, q_1 \succ$ if and only if $p \vee q \leq p_1 \vee q_1$. First let $\prec p, q \succ \subseteq \prec p_1, q_1 \succ$. Then $p_1 \preceq p \preceq q \preceq q_1$ and consequently $p_1 \vee o \leq p \vee o \leq q \vee o \leq q_1 \vee o$. This implies $p \vee q = (q \vee o) \wedge \varphi(p \vee o) \leq (q_1 \vee o) \wedge \varphi(p_1 \vee o) = p_1 \vee q_1$. Conversely let $p \vee q \leq p_1 \vee q_1$. It is $p \leq p \vee o, p \leq p_1 \vee q_1 = (q_1 \vee o) \wedge \varphi(p_1 \vee o) \leq \varphi(p_1 \vee o)$, so that the set $\{p \vee o, \varphi(p_1 \vee o)\}$ has a lower bound and consequently $p_1 \vee o \leq p \vee o$ by (2). Analogously $q \leq p_1 \vee q_1 = (q_1 \vee o) \wedge \varphi(p_1 \vee o) \leq q_1 \vee o,\ q \leq q \vee i = \varphi(q \vee o)$, so that $q \vee o = \varphi^{-1}(\varphi(q \vee o)) \leq q_1 \vee o$. We conclude that $p_1 \preceq p,\ q \preceq q_1$.

Evidently o is the least, i the greatest element of $(A, \preceq)$, so that $(A, \preceq) \in \mathcal{A}_\alpha$. The proof of 1.1. is complete. $\square$

To verify that no of the conditions (1)-(3) can be omitted, let us see the partially ordered sets in Figs. 1-3.

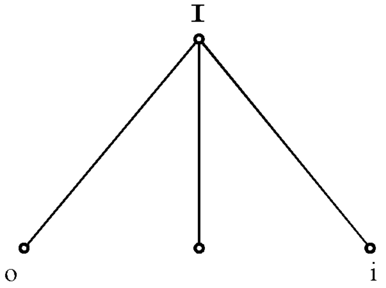

Fig. 1

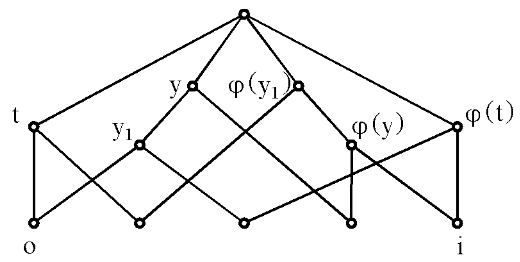

Fig. 2

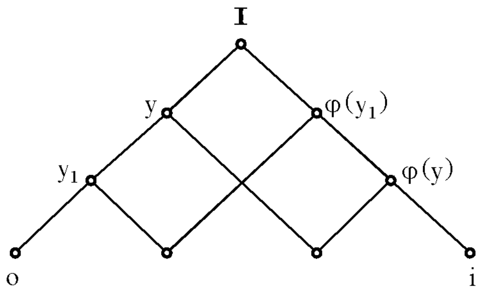

Fig. 3

Each of these partially ordered sets fulfils the condition (β_1) trivially, the i-th partially ordered set fails to satisfy (i), while the other conditions are fulfilled.

If $\mathbb{P}$ is isomorphic to Int $\mathbb{A}$, $\mathbb{A} \in \mathcal{A}_\alpha$, then there exist $o, i \in$ Min $\mathbb{P}$ and φ as in 1.1. A natural question arises if other o_1, i_1, φ_1 satisfying (1)-(3) can exist for the same $\mathbb{P}$. It is easy to see that if o, i, φ satisfy (1)-(3), then also $o_1 = i, i_1 = o, \varphi_1 = \varphi^{-1}$ satisfy (1)-(3). The resulting $\mathbb{A}_1$ is the dual of $\mathbb{A}$. But also other o_1, i_1, φ_1 can exist, as we can see in Fig. 4.

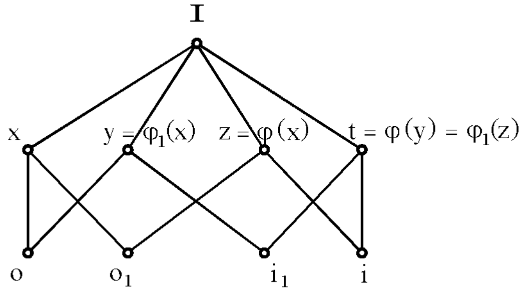

Fig. 4

Notice that the partially ordered set in Fig. 4 is directly reducible, namely it is isomorphic to Int $\mathbb{A}$ $\times$ Int $\mathbb{A}$ for $\mathbb{A}$ being a two-element chain. We will prove that if $\mathbb{P}$ is directly irreducible, such a situation cannot occur. More precisely, we will prove the following theorem.

Theorem 1.2. *Let $\mathbb{P} = (P, \leq)$ be a directly irreducible partially ordered set with the greatest element I and let $\mathbb{P}$ fulfil the condition (β_1). If o, i, φ and o_1, i_1, φ_1 are as in 1.1, then either $o_1 = o, i_1 = i, \varphi_1 = \varphi$ or $o_1 = i, i_1 = o, \varphi_1 = \varphi^{-1}$.*

To prove this we make use of a lemma.

Lemma 1.3. *Let $\mathbb{P} = (P, \leq)$ be a partially ordered set with the greatest element I and let $\mathbb{P}$ fulfil the condition (β_1). Further let o, i, φ be as in 1.1., $p_0 \in$ Min $\mathbb{P}$. Then the interval $<p_0, I>$ is isomorphic to the direct product $<p_0 \vee o, I> \times <p_0 \vee i, I>$.*

Proof. Let us define:

$$\chi : <p_0, I> \to <p_0 \vee o, I> \times <p_0 \vee i, I>$$

by $\chi(x) = (x \vee o, x \vee i)$ for each $x \in P, x \geq p_0$. To show that χ is onto, let $u, v \in P, u \geq p_0 \vee o, v \geq p_0 \vee i$. The set $\{u, v\}$ has a lower bound, so that $\varphi^{-1}(v) \leq u$ and consequently $u \wedge v$ exists. Denote $x = u \wedge v$. Evidently $p_0 \leq x$. In view of B, we have $u = x \vee o, v = x \vee i$ and thus $\chi(x) = (u, v)$. The implication $x \leq x' \Longrightarrow \chi(x) \leq \chi(x')$ si evident, while the opposite one follows from (1). $\square$

Proof of 1.2. In view of Lemma 1.3, the interval $< o_1, I >$ is isomorphic to $< o_1 \vee o, I > \times < o_1 \vee i, I >$. Let $\preceq$ and $\preceq_1$ be the partial order defined in

$A = \text{Min}\,\mathbb{P}$ as in the proof of 1.1 using o, i, φ and o_1, i_1, φ_1, respectively. Then $(A, \preceq_1)$ is isomorphic to $<o_1, I>$. In view of 1.3, the interval $<o_1, I>$ is isomorphic to $<o_1 \vee o, I> \times <o_1 \vee i, I>$. But $(A, \preceq_1)$ is directly irreducible because otherwise $\mathbb{P}$, which is isomorphic to Int $(A, \preceq_1)$, would be also directly reducible. Hence either $o_1 \vee o = I$ or $o_1 \vee i = I$. Suppose, e.g., that the first possibility occurs. Then $o = (o \vee o_1) \wedge (o \vee i_1) = I \wedge (o \vee i_1) = o \vee i_1$ and this implies $i_1 = o$. Further $o_1 = (o_1 \vee o) \wedge (o_1 \vee i) = I \wedge (o_1 \vee i) = o_1 \vee i$, which implies $o_1 = i$. Now let $z \in <o_1, I> = <i, I>$. We will show that $\varphi_1(z) = \varphi^{-1}(z)$. We have $\varphi^{-1}(z) \in <o, I>$ and $\varphi^{-1}(z) \wedge z \in \text{Min}\,\mathbb{P}$ by A. But $z \in <o_1, I>, \varphi^{-1}(z) \in <i_1, I>$, so that using B we obtain $\varphi^{-1}(z) = \varphi_1(z)$. Assuming that $o_1 \vee i = I$ we obtain analogously $o_1 = o, i_1 = i, \varphi_1 = \varphi$. $\square$

2. The class $\mathcal{P}(\mathcal{A}_\beta)$

In this section we will characterize partially ordered sets $\mathbb{P} = (P, \leq)$ belonging to the class $\mathcal{P}(\mathcal{A}_\beta)$. Without loss of generality we can assume that $\mathbb{P}$ is connected. Namely $\mathbb{P}$ is isomorphic to Int $\mathbb{A}$ if and only if each its maximal connected subset $\mathbb{P}_i$ is isomorphic to Int $\mathbb{A}_i$ for some $\mathbb{A}_i$.

Consider the following condition concerning the partially ordered set $\mathbb{P} = (P, \leq)$:

(β) if $x \in P$, then there exist $u \in \text{Min}\,\mathbb{P}, v \in \text{Max}\,\mathbb{P}$ with $u \leq x \leq v$.

If $P_1 \subseteq P$ and for some $x, x_1 \in P_1$ there exists supremum of $\{x, x_1\}$ in P_1 with the inherited order, this element will be denote by $x \vee_{P_1} x_1$. But instead of $x \vee_P x_1$ we will write $x \vee x_1$, as so far.

Theorem 2.1. *Let $\mathbb{P} = (P, \leq)$ be a connected partially ordered set. Then $\mathbb{P}$ belongs to $\mathcal{P}(\mathcal{A}_\beta)$ if and only if $\mathbb{P}$ satisfies (β) and for each $y \in \text{Max}\,\mathbb{P}$ there exist $p_0(y), p_1(y) \in \text{Min}\,\mathbb{P}$ and a dual isomorphism*

$$\varphi_y : <p_0(y), y> \to <p_1(y), y>$$

satisfying the conditions (1)-(3) of 1.1. in $(y>$ and, moreover, it holds:

(4) *if $p \in \text{Min}\,\mathbb{P}, y, z \in \text{Max}\,\mathbb{P}, p \leq y, z$ and $p_i(y) = p_j(z) = q$ for some $i, j \in \{0, 1\}$, then $p \vee_{(y>} q = p \vee_{(z>} q$;*

(5) *if for some $y, y' \in \text{Max}\,\mathbb{P}$ there exists $q \in \text{Min}\,\mathbb{P}, q \leq y, y'$ and the elements $p_0(y), p_1(y), p_0(y'), p_1(y')$ are different, then there exist $z, z' \in \text{Max}\,\mathbb{P}$ and $k \in \{0, 1\}$ such that y, y', z, z' are different, $q \leq z, z'$ and $z = p_0(y) \vee p_k(y'), z' = p_1(y) \vee p_{1-k}(y')$;*

(6) *if $y_1, y_2, \dots, y_n \in \text{Max}\,\mathbb{P} (n \in N)$ and there exist $i_1, \dots, i_n \in \{0, 1\}$ such that $p_{1-i_k}(y_k) = p_{i_{k+1}}(y_{k+1})$ for each $k \in \{1, \dots, n-1\}$ and $p_{1-i_n}(y_n) = p_{i_1}(y_1)$, then n is even.*

First we will show that no of the conditions (4)-(6) can be omitted. If we take $\mathbb{P}$ as in Fig. 5, the condition (4) does not hold, while (5), (6) are satisfied. Fig. 6

shows that (5) does not follow from (4) and (6) and finally $\mathbb{P}$ in Fig. 7 does not satisfy (6), while (4) and (5) hold.

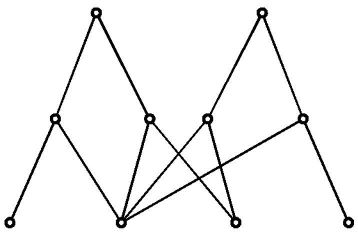

Fig. 5

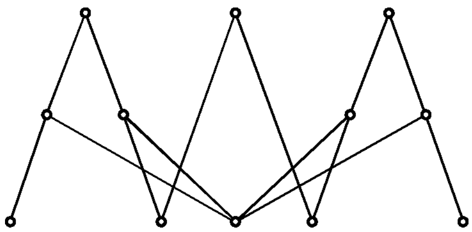

Fig. 6

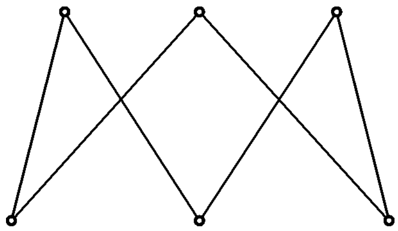

Fig. 7

The proof of 2.1 takes the remaining part of this section.

First let $\mathbb{P} \in \mathcal{P}(\mathcal{A}_\beta)$. Then $\mathbb{P}$ is isomorphic to $\mathrm{Int}\,\mathbb{A}$ for a partially ordered set $\mathbb{A} = (A, \preceq) \in \mathcal{A}_\beta$. Evidently $\mathrm{Int}\,\mathbb{A}$ fulfils (β). If $y = \prec u, v \succ$ is any maximal interval in $\mathbb{A}$ we take $\prec u \succ$ for $p_0(y)$ and $\prec v \succ$ for $p_1(y)$ and we define $\varphi_y(\prec u, w \succ) = \prec w, v \succ$. It is easy to see that (1)-(6) are satisfied. Consequently $\mathbb{P}$ has also the required properties.

Now we are going to prove the converse. So, throughout this section, we will suppose that $\mathbb{P}$ satisfies (β) and for each $y \in \mathrm{Max}\,\mathbb{P}$ there exist $p_0(y), p_1(y) \in \mathrm{Min}\,\mathbb{P}$ and a dual isomorphism $\varphi_y : < p_0(y), y > \to < p_1(y), y >$ satisfying (1)-(3) in $(y>$ and, moreover, (4)-(6) hold.

Lemma 2.2. *If $y, z \in \mathrm{Max}\,\mathbb{P}, p_i(y) \leq z$ for some $i \in \{0,1\}$, then $p_i(y) \in \{p_0(z), p_1(z)\}$.*

Proof. If $y = z$, there is nothing to prove. So let $y \neq z$. Assume that the elements $p_0(y), p_1(y), p_0(z), p_1(z)$ are different. As $p_i(y) \leq y, z$, there exists $y' \in \mathrm{Max}\,\mathbb{P}$ different from z (and from y) such that $y' = p_i(y) \vee p_j(z)$ for some $j \in \{0,1\}$. But then $z \geq p_i(y), p_j(z)$ implies $z \geq y'$, a contradiction. So $p_0(y), p_1(y), p_0(z), p_1(z)$ are not different. Suppose that $p_i(y) \notin \{p_0(z), p_1(z)\}$. Then necessarily $p_{1-i}(y) \in \{p_0(z), p_1(z)\}$. Let, e.g., $p_{1-i}(y) = p_0(z)$. Using (4) we obtain $y = p_i(y) \vee_{(y>} p_{1-i}(y) = p_i(y) \vee_{(z>} p_0(z) \leq z$, a contradiction. Hence $p_i(y) \in \{p_0(z), p_1(z)\}$. $\square$

As an immediate consequence of 2.2 we obtain:

Lemma 2.3. *If $y, y', z \in \operatorname{Max} \mathbb{P}, p_i(y) \neq p_j(y')$ for some $i, j \in \{0,1\}$ and $z \geq p_i(y), p_j(y')$, then one of $p_i(y), p_j(y')$ is $p_0(z)$, the other is $p_1(z)$.*

Lemma 2.4. *Let $y \in \operatorname{Max} \mathbb{P}, p \in \operatorname{Min} \mathbb{P}, p \leq y, i \in \{0,1\}$. Then $p \vee_{(y>} p_i(y) = p \vee p_i(y)$.*

Proof. Let $t \in P, t \geq p, p_i(y)$ and let $z \in \operatorname{Max} \mathbb{P}, z \geq t$. In view of 2.2. we have $p_i(y) = p_j(z)$ for some $j \in \{0,1\}$. Using (4) we obtain $p \vee_{(y>} p_i(y) = p \vee_{(z>} p_i(y) \leq t$. $\square$

As we have remarked in the preceding section, if $y \in \operatorname{Max} \mathbb{P}$, then $y = p_0(y) \vee_{(y>} p_1(y)$. Using 2.4. we obtain:

Corollary 2.5. *If $y \in \operatorname{Max} \mathbb{P}$, then $y = p_0(y) \vee p_1(y)$.*

Lemma 2.6. *Let $y \in \operatorname{Max} \mathbb{P}, p, q \in \operatorname{Min} \mathbb{P}, p, q \leq y$ and $p \vee p_i(y) \leq q \vee p_i(y)$ for some $i \in \{0,1\}$. Then $p \vee_{(y>} q = p \vee q$.*

Proof. Let $t \in P$, $t \geq p, q$. By (β) there exists $z \in \operatorname{Max} \mathbb{P}$ with $z \geq t$.
Distinguish two cases:

(a) $p_i(y) \in \{p_0(z), p_1(z)\}$ or $p_{1-i}(y) \in \{p_0(z), p_1(z)\}$;
(b) all $p_0(y), p_1(y), p_0(z), p_1(z)$ are different.

If $p_i(y) = p_j(z)$ $(j \in \{0,1\})$, we have $p \vee_{(z>} q = (q \vee p_j(z)) \wedge_{(z>} (p \vee p_{1-j}(z)) \leq q \vee p_j(z) = q \vee p_i(y) \leq y$, so that $p \vee_{(y>} q \leq p \vee_{(z>} q \leq t$ by E, B and 2.4. If $p_{1-i}(y) \in \{p_0(z), p_1(z)\}$, we proceed analogously using that $q \vee p_{1-i}(y) \leq p \vee p_{1-i}(y)$. Let us see the case (b). By (5) there exists $y' \in \operatorname{Max} \mathbb{P}, y' = p_i(y) \vee p_j(z)$ for some $j \in \{0,1\}$. Now $p \vee_{(y>} q \leq y', p \vee_{(y'>} q \leq y$, as we have shown above. Hence $p \vee_{(y'>} q = p \vee_{(y>} q$. Analogously $p \vee_{(y'>} q = p \vee_{(z>} q$ and as $p \vee_{(z>} q \leq t$, we have $p \vee_{(y>} q \leq t$, completing the proof. $\square$

If $y, z \in \operatorname{Max} \mathbb{P}, i, j \in \{0,1\}$, by a *zig-zag* connecting $p_i(y)$ with $p_j(z)$ a sequence $p_i(y) = p_{i_1}(z_1), p_{1-i_1}(z_1) = p_{i_2}(z_2), \dots, p_{1-i_{n-1}}(z_{n-1}) = p_{i_n}(z_n), p_{1-i_n}(z_n) = p_j(z)$ with $z_1, \dots, z_n \in \operatorname{Max} \mathbb{P}, i_1, \dots, i_n \in \{0,1\}$ will be meant. The number n will be mentioned as the length of this zig-zag. If, moreover, $p_{i_1}(z_1) = p_{1-i_n}(z_n)$, then we will refer to as a *closed zig-zag* of the length n.

Evidently the condition (6) can be rewritten in such a way that each closed zig-zag is of an even length.

Lemma 2.7. *Let $y, z \in \operatorname{Max} \mathbb{P}, q \in \operatorname{Min} \mathbb{P}, q \leq y, z, i, j \in \{0,1\}$. Then there exists a zig-zag connecting $p_i(y)$ with $p_j(z)$.*

Proof. If $p_0(y), p_1(y), p_0(z), p_1(z)$ are not different, the assertion is trivial. So let $p_0(y), p_1(y), p_0(z), p_1(z)$ are different. Then there exists $y' \in \operatorname{Max} \mathbb{P}$ such that $y' = p_i(y) \vee p_j(z)$ or $y' = p_i(y) \vee p_{1-j}(z)$. In the first case we have $p_i(y) = p_k(y'), p_{1-k}(y') = p_j(z)$ for some $k \in \{0,1\}$ by 2.3. In the latter case we get from $p_i(y)$ to $p_j(z)$ through y' and z. $\square$

Now using connectedness of $\mathbb{P}$, the following assertion can be proved easily.

Lemma 2.8. *Let $y, z \in \operatorname{Max} \mathbb{P}, i, j \in \{0,1\}$. Then there exists a zig-zag connecting $p_i(y)$ with $p_j(z)$.*

Proof. By connectedness of $\mathbb{P}$, there exists a sequence $p_i(y), z_1, q_1, z_2, q_2, ..., q_{n-1}, z_n, p_j(z)$ such that $z_1, ..., z_n \in \operatorname{Max} \mathbb{P}, q_1, ..., q_{n-1} \in \operatorname{Min} \mathbb{P}, p_i(y) \leq z_1, z_n \geq p_j(z), q_k \leq z_k, z_{k+1}$ for each $k \in \{1, ..., n-1\}$. Now we will prove the assertion by induction on n. If $n = 1$, there is nothing to prove. Let the assertion hold for $n = l$ and let $p_i(y), z_1, q_1, ..., q_l, z_{l+1}, p_j(z)$ be a sequence as above. By the induction hypothesis, there exists a zig-zag connecting $p_i(y)$ with $p_0(z_l)$. Using 2.7. we obtain that there exists a zig-zag connecting $p_0(z_l)$ with $p_j(z)$. Connecting both zig-zags we get a zig-zag from $p_i(y)$ to $p_j(z)$. □

Now let us fix any $y_0 \in \operatorname{Max} \mathbb{P}$ and take any of $p_0(y_0), p_1(y_0)$, e.g. $p_0(y_0)$. The condition (6) ensures that for any $y \in \operatorname{Max} \mathbb{P}$ and $i \in \{0,1\}$ each zig-zag connecting $p_0(y_0)$ with $p_i(y)$ is of an even length, or each has an odd length. Moreover, if each zig-zag connecting $p_0(y_0)$ with $p_i(y)$ is of an even length, then each zig-zag connecting $p_0(y_0)$ with $p_{1-i}(y)$ is of an odd length and vice versa. So if we define $\mathcal{M}_0(\mathbb{P}) \subseteq \{p_i(y) : y \in \operatorname{Max} \mathbb{P}, i \in \{0,1\}\}$ by

$p_0(y_0) \in \mathcal{M}_0(\mathbb{P})$,

$p_i(y) \in \mathcal{M}_0(\mathbb{P})$ ($y \in \operatorname{Max} \mathbb{P}, i \in \{0,1\}$) if and only if zig-zags connecting $p_0(y_0)$ with $p_i(y)$ have even lengths,

the set $\mathcal{M}_0(\mathbb{P})$ contains just one of $p_0(y), p_1(y)$ for each $y \in \operatorname{Max} \mathbb{P}$.

Set $A = \operatorname{Min} \mathbb{P}$ and define a relation $\preceq$ in A by

$p \preceq q \iff$ there exists $y \in \operatorname{Max} \mathbb{P}$ such that $p, q \leq y$ and $p \vee p_i(y) \leq q \vee p_i(y)$ for $i \in \{0,1\}$ with $p_i(y) \in \mathcal{M}_0(\mathbb{P})$.

First of all we will prove:

Lemma 2.9. *If $p \preceq q$, then for any $z \in \operatorname{Max} \mathbb{P}$ with $p, q \leq z$ we have $p \vee p_j(z) \leq q \vee p_j(z)$ for $j \in \{0,1\}$ such that $p_j(z) \in \mathcal{M}_0(\mathbb{P})$.*

Proof. If $p \preceq q$, then there exists $y \in \operatorname{Max} \mathbb{P}$ such that $p, q \leq y$ and $p \vee p_i(y) \leq q \vee p_i(y)$ for $i \in \{0,1\}$ with $p_i(y) \in \mathcal{M}_0(\mathbb{P})$. Let $z \in \operatorname{Max} \mathbb{P}, p, q \leq z$ and let $p_j(z) \in \mathcal{M}_0(\mathbb{P})$ ($j \in \{0,1\}$). Distinguish the cases:

$p_i(y) = p_j(z)$;

$p_i(y) \neq p_j(z), p_{1-i}(y) = p_{1-j}(z)$;

$p_i(y) \neq p_j(z), p_{1-j}(y) \neq p_{1-j}(z)$.

In the first case the assertion follows from (4) and 2.4. In the second case we have $p \vee p_{1-j}(z) = p \vee p_{1-i}(y) \geq q \vee p_{1-i}(y) = q \vee p_{1-j}(z)$, so that $p \vee p_j(z) \leq q \vee p_j(z)$. In the third case all $p_0(y), p_1(y), p_0(z), p_1(z)$ are different. According to (5) and using that $p_i(y), p_j(z) \in \mathcal{M}_0(\mathbb{P})$ we obtain that there exists $y' \in \operatorname{Max} \mathbb{P}$ such that $p, q \leq y', y' = p_i(y) \vee p_{1-j}(z)$. In view of 2.3, it is $p_i(y) = p_k(y'), p_{1-j}(z) = p_{1-k}(y')$

for some $k \in \{0,1\}$. Now we have $p \vee p_k(y') = p \vee p_i(y) \leq q \vee p_i(y) = q \vee p_k(y')$ and consequently $p \vee p_{1-k}(y') \geq q \vee p_{1-k}(y')$. But $p \vee p_{1-k}(y') = p \vee p_{1-j}(z), q \vee p_{1-k}(y') = q \vee p_{1-j}(z)$, so that $p \vee p_j(z) \leq q \vee p_j(z)$. □

Lemma 2.10. *If $p \preceq q, q \preceq r$, then there exists $t \in \mathrm{Max}\, \mathbb{P}$ with $p, q, r \leq t$.*

Proof. Let $p \preceq q, q \preceq r$. Then there exist $y, z \in \mathrm{Max}\, \mathbb{P}$ such that $p, q \leq y$, $q, r \leq z$ and if $p_i(y), p_j(z) \in \mathcal{M}_0(\mathbb{P})$, it is $p \vee p_i(y) \leq q \vee p_i(y), q \vee p_j(z) \leq r \vee p_j(z)$. If $p_i(y) = p_j(z)$, we have $p \leq p \vee p_i(y) \leq q \vee p_i(y) = q \vee p_j(z) \leq z$, so that $p, q, r \leq z$. Further let us suppose that $p_i(y) \neq p_j(z), p_{1-i}(y) = p_{1-j}(z)$. Then we have $r \leq r \vee p_{1-j}(z) \leq q \vee p_{1-j}(z) = q \vee p_{1-i}(y) \leq y$, so that $p, q, r \leq y$. Finally if $p_0(y), p_1(y), p_0(z), p_1(z)$ are different, we use (5) and we take $y' = p_i(y) \vee p_{1-j}(z)$. There exists $k \in \{0,1\}$ such that $p_i(y) = p_k(y'), p_{1-j}(z) = p_{1-k}(y')$. Now it is $p \leq p \vee p_i(y) \leq q \vee p_i(y) = q \vee p_k(y') \leq y'$ and proceeding as in the previous case taking y' instead of y we obtain $p, q, r \leq y'$. □

Lemma 2.11. *The relation $\preceq$ is a partial order in A and $(A, \preceq) \in \mathcal{A}_\beta$.*

Proof. The reflexivity is trivial, the antisymmetry follows immediately from 2.9. The transitivity is a consequence of 2.10. and 2.9. To prove $(A, \preceq) \in \mathcal{A}_\beta$ let $p \in A = \mathrm{Min}\, \mathbb{P}$. Take any $y \in \mathrm{Max}\, \mathbb{P}$ with $y \geq p$. If $p_i(y) \in \mathcal{M}_0(\mathbb{P})$, then evidently $p_i(y) \in \mathrm{Min}\,(A, \preceq), p_{1-i}(y) \in \mathrm{Max}\,(A, \preceq)$ and it is $p_i(y) \preceq p \preceq p_{1-i}(y)$. □

Now let us define $\Phi : \mathrm{Int}\ (A, \preceq) \to P$ by

$$\Phi(\prec p, q \succ) = p \vee q \quad (p, q \in A, p \preceq q).$$

Notice that if $p \preceq q$, then $p \vee q$ exists by 2.6.

Lemma 2.12. *The mapping Φ is an isomorphism of* $(\mathrm{Int}\ (A, \preceq), \subseteq)$ *onto* $\mathbb{P} = (P, \leq)$.

Proof. To prove that Φ is onto, let $x \in P$. Take any $y \in \mathrm{Max}\, \mathbb{P}, y \geq x$. In view of the results of the previous section, we have $x = p \vee_{(y>} q$ for some $p, q \in \mathrm{Min}\, \mathbb{P}, p, q \leq y$ with $p \vee_{(y>} p_0(y) = p \vee p_0(y), q \vee_{(y>} p_0(y) = q \vee p_0(y)$ (and hence also $p \vee p_1(y), q \vee p_1(y)$) being comparable. Hence p, q are also comparable. If, e.g., $p \preceq q$, we have $\Phi(\prec p, q \succ) = p \vee q = p \vee_{(y>} q = x$. It remains to show that if $p \preceq q, p_1 \preceq q_1$, then $\prec p, q \succ \subseteq \prec p_1, q_1 \succ$ is equivalent to $p \vee q \leq p_1 \vee q_1$. If $p \vee q \leq p_1 \vee q_1$, we take $y \in \mathrm{Max}\, \mathbb{P}, y \geq p_1 \vee q_1$. Using F we obtain $\prec p, q \succ \subseteq \prec p_1, q_1 \succ$ immediately. Conversely let $\prec p, q \succ \subseteq \prec p_1, q_1 \succ$. Then $p_1 \preceq p \preceq q \preceq q_1$ and using 2.10. we obtain that there exist $y, z \in \mathrm{Max}\, \mathbb{P}$ such that $p_1, p, q_1 \leq y$ and $p_1, q, q_1 \leq z$. Applying F to $(y>$ and $(z>$ we obtain $p \leq p_1 \vee q_1, q \leq p_1 \vee q_1$. Consequently $p \vee q \leq p_1 \vee q_1$. The proof is complete. □

References

[1] Igošin, V. I., *Selfduality of lattices of intervals of finite lattices*, Inst. matem. Sibir. Otdel. AN SSSR, Meždunarodnaja konferencija po algebre posvjaščennaja pamjati A. I. Mal'ceva, Tezisyy dokladov po teoriji modelej i algebraičeskich sistem, Novosibirsk 1989, s. 48.

[2] Igošin, V. I., *Lattices of intervals and lattices of convex sublattices of lattices*, Uporjadočennyje množestva i rešotki. Saratov **6** (1990), 69–76.

[3] Igošin, V. I., *Identities in interval lattices of lattices*, Coll. Math. Soc. J. Bolyai 33 (Contributions to Lattice Theory), Szeged 1980 (1983), 491–501.

[4] Igošin, V. I., *On lattices with restriction on their intervals*, Coll. Math. Soc. J. Bolyai 43 (Lectures in Universal Algebra), Szeged 1983 (1986), 209–216.

[5] Igošin, V. I., *Algebraic characteristic of lattices of intervals*, Uspechi matem. nauk **40** (1985), 205–206.

[6] Igošin, V. I., *Semimodularity in lattices of intervals*, Math. Slovaca **38** (1988), 305–308.

[7] Jakubík, J., *Selfduality of the system of intervals of a partially ordered set*, Czechoslov. Math. J. **41** (1991), 135–140.

[8] Jakubík, J., Lihová, J., *Systems of intervals of partially ordered sets*, Math. Slovaca **46** (1996 No. 4), 355–361.

[9] Kolibiar, M., *Intervals, convex sublattices and subdirect representations of lattices*, Universal Algebra and Applications, Banach Center Publications, Vol. 9, Warsaw 1982, 335–339.

[10] Lihová, J., *Posets having a selfdual interval poset*, Czechoslov. Math. J. **44** (1994), 523–533.

[11] Lihová, J., *On posets with isomorphic interval posets*, Czechoslov. Math. J. **49** (1999), 67–80.

[12] Slavík, V., *On lattices with isomorphic interval lattices*, Czechoslov. Math. J. **35** (1985), 550–554.

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