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A NOTE ON REGULARITY FOR NONLINEAR ELLIPTIC SYSTEMS

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Abstract. The $L^{2,\lambda}$ - regularity of the gradient of weak solutions to nonlinear elliptic systems is proved.

1. Introduction

In this paper we consider the problem of regularity of the first derivatives of weak solutions to the nonlinear elliptic system

$$-D_\alpha a_i^\alpha(x,u,Du) = a_i(x,u,Du)\,, \quad i=1,\dots,N,\ \alpha=1,\dots,n\,, \tag{1}$$

where $a_i^\alpha(x,u,p)$, $a_i(x,u,p)$ are Caratheodorian mappings from $(x,u,p)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}$ into $\mathbb{R}$. A function $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ is called a weak solution to (1) in Ω if

$$\int_\Omega a_i^\alpha(x,u,Du)D_\alpha\varphi^i(x)\,dx = \int_\Omega a_i(x,u,Du)\varphi^i(x)\,dx \quad \forall\,\varphi\in C_0^\infty(\Omega,\mathbb{R}^N)\,.$$

In case of a general system (1) only partial regularity can be expected for $n>2$, see e.g. [Ca], [Gia], [Ne]. Under the assumptions below we will prove $L^{2,\lambda}$ - regularity $(0<\lambda<n)$ of gradient of weak solutions for the system (1) whose coefficients $a_i^\alpha(x,u,Du)$ have the form

$$a_i^\alpha(x,u,Du) = A_{ij}^{\alpha\beta}(x)D_\beta u^j + g_i^\alpha(x,u,Du)\,, \tag{2}$$

where $A_{ij}^{\alpha\beta}$ is a matrix of functions satisfying the following condition of strong ellipticity

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \ge \nu\,|\xi|^2\,, \quad \text{a.e. } x\in\Omega,\ \forall\,\xi\in\mathbb{R}^{nN};\nu>0 \tag{3}$$

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and $g_i^\alpha(x,u,z)$ are smooth functions with sublinear growth in z. In what follows, we formulate the smoothness and the growth conditions for the functions $A_{ij}^{\alpha\beta}(x)$, $g_i^\alpha(x,u,z)$ and $a_i(x,u,z)$ precisely.

In [Da] the first author has proved $L^{2,\lambda}$ - regularity of gradient of weak solutions to (1) in situation when the coefficients $A_{ij}^{\alpha\beta}$ are continuous. In [DV] the authors have shown the analogous result under another assumptions on the coefficients $A_{ij}^{\alpha\beta}$. In [DV] it is supposed that $A_{ij}^{\alpha\beta} \in L^\infty(\Omega) \cap \mathcal{L}_\Phi(\Omega)$, where $\Phi = \Phi(r) = 1/(1+|\ln r|)$. The functions from such class are discontinuous in general, see definition and Proposition 1 below.

In this paper the coefficients $A_{ij}^{\alpha\beta}$ belong to $L^\infty(\Omega) \cap VMO(\Omega)$ (for definition see below) and the result of this paper may be seen as a generalization of that from [DV], see Remark 2 below. The proof of the result is based on method analogous to that in [DV].

2. Notations and definitions

We will consider bounded open set $\Omega \subset \mathbb{R}^n$ with points $x = (x_1, \dots x_n)$, $n \geq 3$ and $u\colon \Omega \to \mathbb{R}^N$, $N \geq 1$, $u(x) = (u^1(x), \dots, u^N(x))$ is a vector-valued function, $Du = (D_1 u, \dots, D_n u)$, $D_\alpha = \partial/\partial x_\alpha$. We will use the convention on summation over repeated indices. The meaning of $\Omega_0 \subset\subset \Omega$ is that the closure of Ω_0 is contained in Ω, i.e. $\overline{\Omega}_0 \subset \Omega$. For the sake of simplicity we denote by $|\cdot|$ the norm in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x \in \mathbb{R}^n$ and r is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n\colon \ | y - x | < r\}$, i.e., the open ball in $\mathbb{R}^n$, $\Omega(x,r) = \Omega \cap B(x,r)$. We denote by $u_{x,r} = |\Omega(x,r)|_n^{-1} \int_{\Omega(x,r)} u(y)\,dy = \fint_{\Omega(x,r)} u(y)\,dy$ the mean value over the set $\Omega(x,r)$ of a function $u \in L^1(\Omega, \mathbb{R}^N)$, where $|\Omega(x,r)|_n$ is the n-dimensional Lebesgue measure of $\Omega(x,r)$. Beside the usually used space $C_0^\infty(\Omega,\mathbb{R}^N)$, the Hölder space $C^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$ and the Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$, $W^{k,p}_{loc}(\Omega,\mathbb{R}^N)$, $W^{k,p}_0(\Omega,\mathbb{R}^N)$ (see, e.g. [KJF]) we introduce the following Morrey spaces.

Definition 1. Let $\lambda \in [0,n]$, $q \in [1,\infty)$. A function $u \in L^q(\Omega,\mathbb{R}^N)$ is said to belong to $L^{q,\lambda}(\Omega,\mathbb{R}^N)$ if

$$||u||^q_{L^{q,\lambda}(\Omega,\mathbb{R}^N)} = \sup\left\{ \frac{1}{r^\lambda} \int_{\Omega(x,r)} |u(y)|^q dy \colon x \in \Omega, r > 0 \right\} < \infty\,,$$

where $\Omega(x,r) = \Omega \cap B_r(x)$.

For more details see [Ca], [Gia], [KJF], [N].

In the next definition we assume that $\Phi\colon [0,d] \to [0,\infty)$ is a continuous, non-decreasing function such that $\sigma \to \Phi(\sigma)/\sigma$ is almost decreasing, i.e. there exists $K_\Phi \geq 1$ such that

$$\frac{K_\Phi \Phi(t)}{t} \geq \frac{\Phi(s)}{s} \quad \forall\ 0 < t < s \leq d\,.$$

Definition 2. A function $u \in L^2(\Omega, \mathbb{R}^N)$ is said to belong to $\mathcal{L}_\Phi(\Omega, \mathbb{R}^N)$ if

$$[u]_{\Phi,\Omega} = \sup \left\{ \frac{1}{\Phi(r)} \left(\fint_{\Omega(x,r)} |u(y) - u_{x,r}|^2 dy \right)^{1/2} : x \in \Omega, r \in (0, \operatorname{diam} \Omega] \right\} < \infty$$

and by $l_\Phi(\Omega, \mathbb{R}^N)$ we denote subspace of all $u \in \mathcal{L}_\Phi(\Omega, \mathbb{R}^N)$ such that

$$[u]_{\Phi,\Omega,r_0} = \sup \left\{ \frac{1}{\Phi(r)} \left(\fint_{\Omega(x,r)} |u(y) - u_{x,r}|^2 dy \right)^{1/2} : x \in \Omega, r \in (0, r_0] \right\} = o(1) \text{ as } r_0 \searrow 0\,.$$

Remark 1. If $\Phi \equiv 1$ we set $\mathcal{L}_\Phi(\Omega, \mathbb{R}^N) \equiv BMO(\Omega, \mathbb{R}^N)$ (bounded mean oscilation) and $l_\Phi(\Omega, \mathbb{R}^N) \equiv VMO(\Omega, \mathbb{R}^N)$ - vanishing mean oscilation, for details see [Ac], [Ca], [Sp].

Some basic properties of above mentioned spaces are formulated in the following properties, for the proofs see [Ac], [Ca], [KJF] and [Sp].

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be a domain of the class $\mathcal{C}^{0,1}$. Then the following assertions holds:*

(i) *$L^{q,n}\left(\Omega, \mathbb{R}^N\right)$ is isomorphic to the $L^\infty(\Omega, \mathbb{R}^N)$.*
(ii) *$u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ and $Du \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN})$, $n - 2 < \lambda < n$ then $u \in C^{0,\alpha}(\Omega, \mathbb{R}^N)$, $\alpha = (\lambda + 2 - n)/2$.*
(iii) *$\mathcal{L}_\Phi(\Omega, \mathbb{R}^N)$ is a Banach space with norm $\|u\|_{\mathcal{L}_\Phi(\Omega,\mathbb{R}^N)} = \|u\|_{L^2(\Omega,\mathbb{R}^N)} + [u]_{\mathcal{L}_\Phi(\Omega,\mathbb{R}^N)}$.*
(iv) *Let $\Phi(r) = 1/(1+|\ln r|)$. Then $C^0(\overline{\Omega}, \mathbb{R}^N) \setminus \mathcal{L}_\Phi(\Omega, \mathbb{R}^N)$ and $(L^\infty(\Omega, \mathbb{R}^N) \cap l_\Phi(\Omega, \mathbb{R}^N)) \setminus C^0(\overline{\Omega}, \mathbb{R}^N)$ are not empty.*
(v) *For $p \in [1, \infty)$, $\Omega' \subset\subset \Omega$, $r_0 \in (0, dist(\Omega', \partial\Omega))$ and $u \in \mathcal{L}_\Phi(\Omega, \mathbb{R}^N)$ set*

$$N_p(u; \Phi, \Omega', r_0) = \sup \left\{ \frac{1}{\Phi(r)} \left(\fint_{\Omega(x,r)} |u(y) - u_{x,r}|^p dy \right)^{1/p} : x \in \Omega', r \in (0, r_0] \right\}.$$

Then we have for each $u \in \mathcal{L}_\Phi(\Omega, \mathbb{R}^N)$

$$N_1(u; \Phi, \Omega', r_0) \le N_p(u; \Phi, \Omega', r_0) \le c(p, n)[u]_{\Phi,\Omega,r_0}\,.$$

Remark 2. It is a trivial fact that $\mathcal{L}_\Phi(\Omega, \mathbb{R}^N) \subseteq VMO(\Omega, \mathbb{R}^N)$ if $\Phi(r)$ vanishes as r approaches zero.

3. Main results

Suppose that for all $(x, u, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ the following conditions hold:

$$|a_i(x,u,z)| \le f_i(x) + L\,|z|^{\gamma_0}\,, \tag{4}$$

$$|g_i^\alpha(x,u,z)| \le f_i^\alpha(x) + L\,|z|^{\gamma}\,, \tag{5}$$

$$g_i^\alpha(x,u,z) z_\alpha^i \ge \nu_1\,|z|^{1+\gamma} - f^2\,, \tag{6}$$

where L, ν_1 are positive constants, $1 \le \gamma_0 < (n+2)/n$, $0 \le \gamma < 1$, f, $f_i^\alpha \in L^{\sigma,\lambda}(\Omega)$, $\sigma > 2$, $0 < \lambda \le n$, $f_i \in L^{\sigma q_0, \lambda q_0}(\Omega)$, $q_0 = n/(n+2)$. We put $A = (A_{ij}^{\alpha\beta})$, $g = (g_i^\alpha)$, $a = (a_i)$, $\widetilde{f} = (f_i)$, $\widetilde{\widetilde{f}} = (f_i^\alpha)$.

Theorem. *Let $u \in W_{loc}^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1) and the conditions (2), (3), (4), (5) and (6) be satisfied. Suppose further that $A_{ij}^{\alpha\beta} \in L^\infty(\Omega) \cap VMO(\Omega)$, $i, j = 1, \dots, N$, $\alpha, \beta = 1, \dots, n$. Then*

$$Du \in \begin{cases} L_{loc}^{2,\lambda}\left(\Omega, R^{nN}\right) & \text{if} \quad \lambda < n \\ L_{loc}^{2,\lambda'}\left(\Omega, R^{nN}\right) \quad \text{with arbitrary} \quad \lambda' < n & \text{if} \quad \lambda = n\,. \end{cases}$$

Corollary. *Let the assumptions of theorem be satisfied. Then*

$$u \in \begin{cases} C^{0,(\lambda-n+2)/2}\left(\Omega, R^N\right) & \text{if} \quad n-2 < \lambda < n \\ C^{0,\gamma}\left(\Omega, R^N\right) \quad \text{with arbitrary} \quad \gamma < 1 & \text{if} \quad \lambda = n\,. \end{cases}$$

Proof. It follows from Poincaré's inequality and Proposition 1(ii). □

4. Auxiliary lemmas

In this section we present the results needed for the proof of Theorem. In $B(x,r) \subset \mathbb{R}^n$ we consider a linear elliptic system

$$-D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = 0 \tag{7}$$

with constant coefficients satisfying (3).

Lemma 1 ([Ca] pp. 54-55)**.** *Let $u \in W^{1,2}(B(x,r), \mathbb{R}^N)$ be a weak solution to the system (7). Then for each $t \in [0,1]$*

$$\int_{B_{tr}} |Du(y)|^2\, dy \le c\, t^n \int_{B_r} |Du(y)|^2\, dy\,.$$

holds.

Lemma 2 ([KN]). *Let $\Phi = \Phi(R)$, $R \in (0,d]$, $d > 0$ be a nonnegative function and let A, B, C, a, b be nonnegative constants. Suppose that for all $t \in (0,1]$ and all $R \in (0,d]$*

$$\Phi(tR) \leq (At^a + B)\Phi(R) + CR^b$$

holds. Futher let $K \in (0,1)$ be such that $\varepsilon = AK^{a-b} + BK^{-b} < 1$. Then

$$\Phi(R) \leq cR^b, \quad R \in (0,d],$$

where $c = \max\{C/K(1-\varepsilon), \sup_{R\in[K\,d,d]} \Phi(R)/R^b\}$.

The following Lemma is the special case of Lemma 3.4 of the paper [Da].

Lemma 3 ([Da], pp.757-758). *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$, $Du \in L^{2,\tau}(\Omega,\mathbb{R}^{nN})$, $0 \leq \tau < n$ and (4) and (5) are satisfied with $f_i \in L^{2q_0,\lambda q_0}(\Omega)$, $f_i^\alpha \in L^{2,\lambda}(\Omega)$, $0 < \lambda \leq n$.*

(i) *Then $a_i \in L^{2q_0,\lambda_0}(\Omega)$ and for each ball $B_R(x) \subset \Omega$ we have*

$$\int_{B_R(x)} |a_i(x,u,Du)|^{2q_0}\,dy \leq c\,R^{\lambda_0}, \tag{8}$$

where $c = c(n, L, \gamma_0, \operatorname{diam}\Omega, \|\widetilde{f}\|_{L^{2q_0,\lambda q_0}(\Omega,\mathbb{R}^N)}, \|Du\|_{c(L)(\Omega,\mathbb{R}^N)})$ and $\lambda_0 = \min\{\lambda q_0, n-(n-\tau)q_0\gamma_0\}$.

(ii) *For each $\varepsilon \in (0,1)$ and all $B_R(x) \subset \Omega$*

$$\int_{B_R(x)} |g_i^\alpha(x,u,Du)|^2\,dy \leq c(L)\,\varepsilon \int_{B_R(x)} |Du|^2\,dy + c\,R^{\lambda_1}. \tag{9}$$

Here $c = c(L,\varepsilon,\gamma,\operatorname{diam}\Omega, \|\widetilde{\widetilde{f}}\|_{L^{2,\lambda}(\Omega,\mathbb{R}^{nN})}, \|Du\|_{L^2(\Omega,\mathbb{R}^N)})$, $\lambda_1 = \lambda$ for $\lambda < n$ and $\lambda_1 < n$ for $\lambda = n$.

Proof. For the proof (i) see [Ca], pp.106-107. According to (5) we have

$$\int_{B_R(x)} |g_i^\alpha(y,u,Du)|^2\,dy \leq c\left(\|\widetilde{\widetilde{f}}\|^2_{L^{2,\lambda}(\Omega,\mathbb{R}^{nN})}\,R^\lambda + \int_{B_R(x)} |Du|^{2\gamma}\,dy\right).$$

Applying the Young inequality we obtain

$$\int_{B_R(x)} |Du|^{2\gamma}\,dy \leq \varepsilon \int_{B_R(x)} |Du|^2\,dy + c(n,\varepsilon,\gamma)R^n$$

for each $\varepsilon \in (0,1)$ and (9) easily follows. $\square$

In the following considerations we will use a result about higher integrability of gradient of weak solution to the system (1).

Proposition 4 ([Gia], p.138). *Suppose that* (2) - (6) *are fulfilled and let* $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ *be a weak solutions of* (1). *Then there exists an exponent* $r > 2$ *such that* $u \in W^{1,r}_{loc}(\Omega, \mathbb{R}^N)$. *Moreover there exists a constant* $c = c(\nu, \nu_1, L, \|A\|_\infty)$ *and* $\widetilde{R} > 0$ *such that for all balls* $B_R(x) \subset \Omega$, $R < \widetilde{R}$ *the following inequality is satisfied*

$$\left(\fint_{B_{R/2}(x)} |Du|^r \, dy \right)^{1/r}$$
$$\le c \left\{ \left(\fint_{B_R(x)} |Du|^2) \, dy \right)^{1/2} + \left(\fint_{B_R(x)} (|f|^r + |\widetilde{\widetilde{f}}|^r) \, dy \right)^{1/r} + R \left(\fint_{B_R(x)} |\widetilde{f}|^{rq_0} dy \right)^{1/rq_0} \right\} .$$

5. Proof of the Theorem

Let $B_{R/2}(x_0) \subset B_R(x_0) \subset \Omega$ be an arbitrary ball and let $w \in W^{1,2}_0(B_{R/2}(x_0), \mathbb{R}^N)$ be a solution of the following system

$$\begin{aligned}(10) \quad & \int_{B_{R/2}(x_0)} (A^{\alpha\beta}_{ij})_{x_0,R/2} D_\beta w^j D_\alpha \varphi^i \, dx \\ &= \int_{B_{R/2}(x_0)} \Big((A^{\alpha\beta}_{ij})_{x_0,R/2} - A^{\alpha\beta}_{ij}(x) \Big) D_\beta u^j D_\alpha \varphi^i \, dx \\ &\quad - \int_{B_{R/2}(x_0)} g^\alpha_i(x, u, Du) D_\alpha \varphi^i \, dx + \int_{B_{R/2}(x_0)} a_i(x, u, Du) \varphi^i \, dx\end{aligned}$$

for all $\varphi \in W^{1,2}_0(B_{R/2}(x_0), \mathbb{R}^N)$. It is known that under the assumption of Theorem such solution exists and is unique for all $R < R'$ (R' is sufficiently small).

Substituting $\varphi = w$ in (10) and using the ellipticity, the Hölder and the Sobolev inequalities we get

$$\begin{aligned}\nu^2 \int_{B_{R/2}(x_0)} |Dw|^2 \, dx \le c \Bigg(& \int_{B_{R/2}(x_0)} |A_{x_0,R/2} - A(x)|^2 |Du|^2 \, dx \\ & + \int_{B_{R/2}(x_0)} |g(x, u, Du)|^2 \, dx + \left(\int_{B_{R/2}(x_0)} |a(x, u, Du)|^{2q_0} \, dx \right)^{1/q_0} \Bigg) \\ = c(I + II + III) \, . &\end{aligned}$$

Taking into account the properties of matrix $A = (A^{\alpha\beta}_{ij})$, Proposition 1(v), Proposition 4 with $r > 2$ and the Hölder inequality ($r' = r/(r-2)$) we obtain

$$
\begin{aligned}
I &\le \left(\int_{B_{R/2}(x_0)} |A(x) - A_{x_0,R/2}|^{2r'}\,dx\right)^{1/r'} \left(\int_{B_{R/2}(x_0)} |Du|^r\,dx\right)^{2/r} \\
&\le c\,R^{n/r'} \left(\fint_{B_{R/2}(x_0)} |A(x) - A_{x_0,R/2}|^{2r'}\,dx\right)^{1/r'} \left(\int_{B_{R/2}(x_0)} |Du|^r\,dx\right)^{2/r} \\
&\le N_{2r'}(A;1,B_{R/2}(x_0),R/2)R^{n/r'} \left(\int_{B_{R/2}(x_0)} |Du|^r\,dx\right)^{2/r} \\
&\le c_1(n,r,R)R^{n/r'} \left(\int_{B_{R/2}(x_0)} |Du|^r\,dx\right)^{2/r},
\end{aligned}
$$

where $c_1 = c_1(n,r,R)$ vanishes as R approaches zero, because $A_{ij}^{\alpha\beta} \in VMO(\Omega)$ (in this step we apply the more generaly condition on $A_{ij}^{\alpha\beta}$ than in [DV]).

To the estimate the last integral in above inequality we use Proposition 4 and we get

$$
\begin{aligned}
&\left(\int_{B_{R/2}(x_0)}\right).\,(.|Du|^r\,dx)^{2/r} \\
&\qquad \le c_2 \left\{ \frac{1}{R^{n(1-2/r)}} \int_{B_R(x)} |Du|^2\,dy + \left(\int_{B_R(x)} (|f|^r + |\tilde{\tilde{f}}|^r)\,dy\right)^{2/r} \right. \\
&\qquad\qquad \left. + R^{2(1-2/r)} \left(\int_{B_R(x)} |\tilde{f}|^{rq_0}\,dy\right)^{2/rq_0} \right\} \\
&\qquad \le c_3 \left(\frac{1}{R^{n(1-2/r)}} \int_{B_R(x)} |Du|^2\,dy + R^{2\lambda/r} + R^{2(r-2+\lambda)/r)} \right),
\end{aligned}
$$

where $c_3 = c_3(r, \|f\|_{L^{r,\lambda}(\Omega)}, \|\tilde{\tilde{f}}\|_{L^{r,\lambda}(\Omega)}, \|\tilde{f}\|_{L^{rq_0,\lambda q_0}(\Omega)})$.

$$
I \le c_4(R) \int_{B_R(x_0)} |Du|^2 dx + c_5 \left(R^{2\lambda/r} + R^{2(r-2+\lambda)/r)}\right) R^{n/r'},
$$

where $c_4(R)$ vanishes as R approaches zero.

We can estimate II and III by means of Lemma 3 (with $\tau = 0$) and we have

$$
\nu^2 \int_{B_{R/2}(x_0)} |Dw|^2\,dx \le c_6 \left\{ (\varepsilon + c_4(R)) \int_{B_R(x_0)} |Du|^2\,dx + R^\mu \right\}, \tag{11}
$$

where $\mu = \min\{(2\lambda + n(r-2))/r, (2\lambda + (n+2)(r-2))/r, \lambda, n + 2 - n\gamma_0\} = \min\{\lambda, n + 2 - n\gamma_0\}$ because $r > 2$.

The function $v = u - w \in W^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ is the solution of the system

$$\int_{B_{R/2}(x_0)} (A_{ij}^{\alpha\beta})_{x_0,R/2} D_\beta v^j D_\alpha \varphi^i \, dx = 0, \qquad \forall \varphi \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)\,. \tag{12}$$

From lemma 1 we have for $t \in (0,1]$

$$\int_{B_{tR/2}(x_0)} |Dv(y)|^2 \, dy \le c_7 \, t^n \int_{B_{R/2}(x_0)} |Dv(y)|^2 \, dy\,.$$

By means of (11) and (12) we obtain for $t \in (0,1]$ and $\varepsilon \in (0,1)$

$$\int_{B_{tR/2}(x_0)} |Du|^2 dx \le c_8 \left\{ (t^n + \varepsilon + c_4(R)) \int_{B_R(x_0)} |Du|^2 \, dx + R^\mu \right\}\,.$$

For $t \in [1,2]$ the above inequality is trivial and we obtain

$$\int_{B_{tR}(x_0)} |Du|^2 \, dx \le c_9 \, (t^n + \varepsilon + c_4(R)) \int_{B_R(x_0)} |Du|^2 \, dx + c_{10} \, R^\mu\,, \qquad \forall \, t \in [0,1] \tag{13}$$

where the constants c_9 and c_{10} depends only on above mentioned parametrs.

Now from Lemma 2 we get the result the following manner. We put $\Phi(R) = \int_{B_R(x_0)} |Du|^2 \, dx$, $A = c_9$, $B = c_9(\varepsilon + c_4(R))$ and $C = c_{10}$. We can choose $0 < K < 1$ such that $AK^{n-\lambda} < 1/2$ (in the case $\lambda = n$ we have $AK^{n-\lambda_1} < 1/2$, where λ_1 is from Lemma 3(ii)). It is obvious that the constants $\varepsilon_0 > 0$, $R_0 > 0$ exist such that $BK^{-\lambda} < 1/2$ ($B = \varepsilon_0 + c_4(R_0)$) and then for all $t \in (0,1)$, $R < R_0$ the assumptions of Lemma 2 are satisfied and therefore

$$\int_{B_R(x_0)} |Du|^2 \, dx \le c \, R^\mu\,.$$

If $\mu = \lambda$ the Theorem is proved. If $\mu < \lambda$ the previous procedure can be repeated with $\tau = \mu$ in Lemma 3. It is clear that after a finite number of steps (since μ increases in each step as it follows from Lemma 3) we obtain $\mu = \lambda$.

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