ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 159 – 169

AN EXISTENCE RESULT FOR FIRST ORDER INITIAL VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. In this paper, a nonlinear alternative for multivalued maps is used to investigate the existence of solutions of first order impulsive initial value problem for differential inclusions in Banach spaces.

1. Introduction

The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Differential equations involving impulse effects occurs in many applications: physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. The reader can see for instance [2], [7], [10], [12], [13], [14], [16], [17] and [21]. However very few results are available for impulsive differential inclusions or related topics see for instance [4], [6], [8], [18], and [19].

The fundamental tools used in the existence proofs of all above mentioned works are essentially fixed point arguments, degree theory, topological transversality or the monotone method combined with upper and lower solutions.

In this paper, we shall be concerned with the existence of solutions of the first order initial value problem for the impulsive differential inclusion:

$$y' \in F(t,y)\,, \quad t \in J\,, \quad t \neq t_k\,, \quad k = 1, \dots, m \tag{1.1}$$

$$y(0) = y_0\,, \tag{1.2}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k))\,, \quad k = 1, \dots, m\,, \tag{1.3}$$

where $F : J \times E \longrightarrow 2^E$ is a bounded, closed and convex multivalued map, $J = [0,T]$ $(0 < T < \infty)$, $y_0 \in E, 0 = t_0 < t_1 < \dots, < t_m < t_{m+1} = T$; $I_k \in$

2000 *Mathematics Subject Classification*: 34A37, 34A60, 34G20.
Key words and phrases: impulsive initial value problem, multivalued map, a priori bounds, existence, fixed point.
Received March 18, 1999.

$C(E,E)$ $(k=1,2...,m)$; and E a real Banach space with the norm $|.|$.
$\Delta y|_{t=t_k}$ denotes the jump of $y(t)$ at $t=t_k$, i.e.,

$$\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)\,,$$

where $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t=t_k$ respectively.

In this paper we shall generalize a recent result for the problem (1.1), (1.2) without impulse effect (i.e. $I_k \equiv 0$, for each $k=1,...,m$) considered by the first author (see [3]). We also generalize to impusive differential inclusions the problem (1.1)-(1.2)-(1.3) considered by Frigon and O'Regan [7] in the single-valued case. Our approach is based on a nonlinear alternative for multivalued maps due to O'Regan ([15]).

2. Preliminaries

In this section, we introduce notations, definitions, and results which are used throughout the paper.
$[a,b]$ denotes a real compact interval of $\mathbb{R}$.
$AC([a,b],E)$ is the Banach space of absolutely continuous functions defined on $[a,b]$ with values in E.
$C([a,b],E)$ is the Banach space of continuous functions from $[a,b]$ into E with norm

$$\|y\|_\infty = \sup\{|y(t)| : t\in[a,b]\} \quad \text{for all} \quad y\in C([a,b],E)\,.$$

Let $y:[a,b]\longrightarrow E$ be measurable function. By $\int_a^b y(s)ds$, we mean the Bochner integral of y, assuming it exists. A measurable function $y:[a,b]\longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral see [20].
$L^1([a,b],E)$ denotes the Banach space of functions Bochner integrable normed by

$$\|y\|_{L^1} = \int_a^b |y(t)|dt \quad \text{for all} \quad y\in L^1([a,b],E)\,.$$

Let $(X,\|.\|)$ be a Banach space. A multivalued map $G:X\longrightarrow 2^X$ has convex (closed) values if $G(x)$ is convex (closed) for all $x\in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X (i.e. $\sup\limits_{x\in B}\{\sup\{\|y\| : y\in G(x)\}\}<\infty$).
G is called upper semicontinuous (u.s.c.) on X if for each $x_0\in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M)\subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B\subseteq X$.

If the multivalued G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n\longrightarrow x_0$, $y_n\longrightarrow y_*$, $y_n\in Gx_n$ imply $y_*\in Gx_0$). G has a fixed point if there is $x\in X$ such that $x\in Gx$.

In the following $BCC(E)$ denotes the set of all nonempty bounded, closed, convex subsets of E.

A multivalued map $G : [a,b] \longrightarrow BCC(X)$ is said to be measurable if for each $x \in X$ the distance between x and $G(t)$ is a measurable function on $[a,b]$.

A multivalued map $F : [a,b] \times E \longrightarrow 2^E$ is said to be an L^1-Carathéodory if

(i) $t \longmapsto F(t,y)$ is measurable for each $y \in E$;

(ii) $y \longmapsto F(t,y)$ is upper semicontinuous for almost all $t \in [a,b]$;

(iii) For each $k > 0$, there exists $h_k \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leq h_k(t)$$

for all $|y| \leq k$ and for almost all $t \in [a,b]$.

For more details on multivalued maps see [1] and [5].

Let $J' = J \backslash \{t_1, ..., t_m\}$, $\Omega = \{y : J \longrightarrow E : y$ is continuous for $t \neq t_k$, $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k) = y(t_k^-)$, $k = 1, ..., m\}$.
$\Omega^1 = \{y \in \Omega : y$ is differentiable almost everywhere on J' and $y' \in L^1(J', E)\}$.
Evidently, Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t \in J} |y(t)| .$$

We shall refer to problem (1.1), (1.2), (1.3) as (NP). By a solution to (NP), we mean a function $y \in \Omega_0^1 := \{y \in \Omega^1 : y(0) = y_0\}$ that satisfies the differential inclusion

$$y'(t) \in F(t, y(t)) \quad \text{almost everywhere on } J' ,$$

and for each $k = 1, ..., m$ the equation

$$\Delta y|_{t=t_k} = I_k(y(t_k)) .$$

The following lemmas are crucial in the proof of our main theorem:

Lemma 2.1. [11] *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying the Carathéodory conditions with the set of L^1-selections S_F is nonempty, and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow BCC(C(I,X)) , \ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.2. (Nonlinear Alternative [15]**)** *Let X be a Banach space with $C \subset X$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $G : \overline{U} \longrightarrow 2^C$ is given by $G = G_1 + G_2$, where $G_1 : \overline{U} \longrightarrow 2^C$ is a compact multivalued map, u.s.c. with convex closed values and the single-valued operator $G_2 : \overline{U} \longrightarrow C$ is a nonlinear contraction (i.e., there exists a continuous nondecreasing function $\phi : [0, \infty) \longrightarrow [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|G_2(x) - G_2(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in \overline{U}$). Then either,*

(i) *G has a fixed point in $\overline{U}$; or*

(ii) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

Remark 2.1. *By* $\overline{U}$ *and* ∂U *we denote the closure of* U *and the boundary of* U *respectively.*

Let us introduce the following hypotheses:

(H1) $F : J \times E \longrightarrow BCC(E)$ has the decomposition $F(t,y) = F_1(t,y) + f_1(t,y)$ with $F_1 : J \times E \longrightarrow BCC(E)$ is an L^1- Carathéodory multivalued map, $f_1 : J \times E \longrightarrow E$ a Carathéodory map and for each fixed $y \in C(J,E)$ the set

$$S_{F_1,y} = \{v \in L^1(J,E) : \ v(t) \in F_1(t,y(t)) \text{ for a.e. } \ t \in J\}$$

is nonempty;

(H2) there exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,y)\| := \sup\{|v| : v \in F(t,y)\} \le p(t)\psi(|y|) \quad \text{for a.e. } \ t \in J \ \text{ and each } \ y \in E;$$

(H3)

$$\int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{du}{\psi(u)}\,, \quad k = 1, ..., m+1\,;$$

Here $N_0 = |y_0|$ and for $k = 2, ..., m+1$ we have

$$N_{k-1} = \sup_{y \in [-M_{k-2}, M_{k-2}]} |I_{k-1}(y)| + M_{k-2}\,, \quad M_{k-2} = \Gamma_{k-1}^{-1}\Big(\int_{t_{k-2}}^{t_{k-1}} p(s)ds\Big)$$

with

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{du}{\psi(u)}\,, \quad z \ge N_{l-1}, \quad l \in \{1, ..., m+1\}\,.$$

(H4) for each bounded set $B \subseteq C(J,E)$, and $t \in J$ the set

$$\Big\{y_0 + \int_0^t v(s)ds : v \in S_{F_1,B}\Big\}$$

is relatively compact where $S_{F_1,B} = \{S_{F_1,y} : y \in B\}$;

(H5) there exist positive constants a_k $(k = 1, ..., m)$, $q \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that

$$|f_1(t,y_1) - f_1(t,y_2)| \le q(t)\phi(|y_1 - y_2|) \ \text{ for a.e. } t \in J \ \text{ and each } y_1, y_2 \in E\,,$$

and

$$|I_k(y_1) - I_k(y_2)| \le a_k|y_1 - y_2| \ \text{ for each } \ y_1, y_2 \in E \ (k = 1, ..., m)$$

with

$$\|q\|_{L^1} + \sum_{k=1}^{k=m} a_k < 1\,.$$

Remark 2.2. If dim $E < \infty$, then $S_{F_1,y} \neq \emptyset$ for any $y \in C(J,E)$ (see [11]).

We have the following auxiliary result

Lemma 2.3. *If $y \in \Omega^1$, then*

$$y(t) = y(0) + \int_0^t y'(s)ds + \sum_{0<t_k<t} [y(t_k^+) - y(t_k)] \,, \quad \textit{for} \quad t \in J\,. \tag{2.1}$$

Proof. Assume that $t_k < t \leq t_{k+1}$ (here $t_0 = 0, t_{m+1} = T$). Then

$$y(t_1) - y(0) = \int_0^{t_1} y'(s)ds\,,$$

$$y(t_2) - y(t_1^+) = \int_{t_1}^{t_2} y'(s)ds\,,$$

$$\ldots$$

$$y(t_k) - y(t_{k-1}^+) = \int_{t_{k-1}}^{t_k} y'(s)ds\,,$$

$$y(t) - y(t_k^+) = \int_{t_k}^{t} y'(s)ds\,.$$

Adding these together, we get

$$y(t) - y(0) - \sum_{i=1}^{i=k}[y(t_i^+) - y(t_i)] = \int_0^t y'(s)ds\,,$$

i.e. Eq. (2.1) hold.

3. Main Result

Theorem 3.1. (A priori bounds on solutions).
Let $y \in \Omega^1$ be a (possible) solution to (NP). Then for each $k = 1, ..., m+1$ there exists a constant M_{k-1} such that

$$\sup\{|y(t)| : t \in [t_{k-1}, t_k]\} \leq M_{k-1}\,.$$

Proof. Let y be a (possible) solution to (NP). Then $y\Big|_{[0,t_1]}$ is a solution to

$$y'(t) \in F(t, y(t)) \quad \text{for all } t \in [0, t_1]\,, \quad y(0) = y_0\,.$$

Then, it is clear that

$$y(t) = y_0 + \int_0^t v(s)\,ds \qquad \text{for } v \in F(t,y) \text{ and } \quad t \in [0,t_1]\,.$$

Set
$$J^0 = \{t \in [0, t_1] : |y(t)| > |y_0|\} .$$
Let $t^* \in [0, t_1]$ be such that
$$||y||_\infty = \max\{|y(t)| : \ t \in [0, t_1]\} = |y(t^*)| .$$
If $t^* \notin J^0$ then $y(t) \equiv y_0$. Otherwise, since $y(0) = y_0$, there exists $\alpha \in (0, t^*)$ such that $|y(\alpha)| = |y_0|$ and $|y(t)| > |y_0|$ for all $t \in (\alpha, t^*]$.
From condition (H2) we infer that
$$|y(t)| \leq |y_0| + \int_0^t p(s)\psi(|y(s)|)ds \ \text{ for a.e. } \ t \in (\alpha, t^*] .$$
Let
$$u\,(t) = |y_0| + \int_0^t p(s)\psi(|y(s)|)ds \ \text{ for a.e. } \ t \in (\alpha, t^*] .$$
Comparing these last two inequalities we see that
$$|y(t)| \leq u\,(t) \quad \text{for all } \ t \in (\alpha, t^*] .$$
Now,
$$u'\,(t) = p(t)\psi(|y(t)|) \ \text{ for a.e. } \ t \in (\alpha, t^*)$$
Since ψ is nondecreasing we have
$$u'\,(t) \leq p(t)\psi(|u(t)|) \ \text{ for a.e. } \ t \in (\alpha, t^*) .$$
From this inequality, it follows that
$$\int_\alpha^{t^*} \frac{u'(t)}{\psi(u(t))}dt \leq \int_\alpha^{t^*} p(s)ds .$$
Using the change of variables formula (see [9]), we get
$$\Gamma_1(u(t^*)) = \int_{|y_0|}^{u(t^*)} \frac{d\sigma}{\psi(\sigma)} \leq \int_\alpha^{t^*} p(s)ds \leq \int_0^{t_1} p(s)ds .$$
In view of (H3), we obtain
$$u(t^*) \leq \Gamma_1^{-1}\Big(\int_0^{t_1} p(s)ds\Big) .$$
Since
$$|y(t)| \leq u\,(t) \quad \text{for all } \ t \in (\alpha, t^*]$$
it follows that
$$|y(t^*)| \leq \Gamma_1^{-1}\Big(\int_0^{t_1} p(s)ds\Big) := M_0 .$$
Therefore
$$\sup_{t\in[0,t_1]} |y(t)| \leq M_0 .$$
Now, $y\Big|_{[t_1,t_2]}$ is a solution to
$$\begin{cases} y'\,(t) \in F(t, y(t)) & \text{for all } t \in [t_1, t_2] \\ \Delta y|_{t=t_1} = I_1(y(t_1)) . \end{cases}$$

Note that

$$|y(t_1^+)| \leq \sup_{y\in[-M_0,+M_0]} |I_1(y)| + \sup_{t\in[0,t_1]} |y(t)|$$

$$\leq \sup_{y\in[-M_0,+M_0]} |I_1(y)| + M_0 := N_1 \,.$$

Suppose $|y(t)| > N_1$ for some $t \in (t_1, t_2]$. Then there exists $\eta \in [t_1, t)$ with $|y(s)| > N_1$ for $s \in (\eta, t]$ and $|y(\eta)| = N_1$. Thus we have

$$|y(s)| \leq N_1 + \int_\eta^s p(\tau)\psi(|y(\tau)|)d\tau \ \text{ for a.e. } \ s \in (\eta, t)\,.$$

Proceeding as above we obtain

$$\Gamma_2(|y(t)|) = \int_{N_1}^{|y(t)|} \frac{d\sigma}{\psi(\sigma)} \leq \int_\eta^t p(s)ds \leq \int_{t_1}^{t_2} p(s)ds\,.$$

This yields

$$\sup_{t\in[t_1,t_2]} |y(t)| \leq \Gamma_2^{-1}\Big(\int_{t_1}^{t_2} p(s)ds\Big) := M_1\,.$$

We continue this process and taking into account that $y\Big|_{[t_m,T]}$ is a solution to the problem

$$\begin{cases} y'(t) \in F(t, y(t)) & \text{for all } t \in [t_m, T] \\ \Delta y|_{t=t_m} = I_m(y(t_m))\,. \end{cases}$$

We obtain that there exists a constant M_m such that

$$\sup_{t\in[t_m,T]} |y(t)| \leq \Gamma_{m+1}^{-1}\Big(\int_{t_m}^{T} p(s)ds\Big) := M_m\,.$$

Consequently, for each possible solution y to (NP) we have

$$\|y\|_\Omega \leq \max\{M_{k-1} : k = 1, ...m+1\} := b\,. \qquad \square$$

Now, we are in a position to state and prove our main result.

Theorem 3.2. *Suppose that hypotheses (H1)-(H5) are satisfied. Then the impulsive initial value problem (NP) has at least one solution.*

Proof. A solution of the problem (NP) is a fixed point for the operator $G : \Omega \longrightarrow 2^\Omega$ defined by

$$Gy := \Big\{h \in \Omega : h(t) = y_0 + \int_0^t v(s)ds + \sum_{0<t_k<t} I_k(y(t_k)) \ : \ v \in S_{F,y}\Big\}$$

where

$$S_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}\,.$$

$S_{F,y}$ can be written as

$$S_{F,y} = S_{F_1,y} + f_1(., y(.))\,.$$

Hence $Gy = G_1y + G_2y$ where

$$G_1y := \left\{ h \in \Omega : h(t) = y_0 + \int_0^t v(s)ds \ : \ v \in S_{F_1,y} \right\}$$

and

$$(G_2y)(t) := \int_0^t f_1(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)) \, .$$

We shall show that G_1 is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: G_1y is convex for each $y \in \Omega$.
Indeed, if h, $\overline{h}$ belong to G_1y, then there exist $v \in S_{F_1,y}$ and $\overline{v} \in S_{F_1,y}$ such that for $t \in J$ we have

$$h(t) = y_0 + \int_0^t v(s)ds$$

and

$$\overline{h}(t) = y_0 + \int_0^t \overline{v}(s)ds \, .$$

Let $0 \leq l \leq 1$. Then for each $t \in J$ we have

$$[lh + (1-l)\overline{h}](t) = y_0 + \int_0^t [lv(s) + (1-l)\overline{v}(s)]ds \, .$$

Since $S_{F_1,y}$ is convex (because F_1 has convex values) then

$$lh + (1-l)\overline{h} \in G_1y \, .$$

Step 2: G_1 sends bounded sets into bounded sets in Ω.
Let $B_R := \{y \in \Omega : \|y\|_\Omega \leq R\}$ be a bounded set in Ω and $y \in B_R$, then for each $h \in G_1y$ there exists $v \in S_{F_1,y}$ such that for each $t \in J$ we have

$$h(t) = y_0 + \int_0^t v(s)ds \, .$$

Thus for each $t \in J$ we get

$$\begin{aligned} |h(t)| &\leq |y_0| + \int_0^t |v(s)|ds \\ &\leq |y_0| + \int_0^t h_R(s)ds \\ &\leq |y_0| + \|h_R\|_{L^1} \, . \end{aligned}$$

Step 3: G_1 sends bounded sets in Ω into equicontinuous sets.

Let $u_1, u_2 \in J,\ u_1 < u_2,\ B_R := \{y \in \Omega : \|y\|_\Omega \leq R\}$ be a bounded set in Ω and $y \in B_R$. For each $h \in G_1 y$ there exists $v \in S_{F_1,y}$ such that for $t \in J$

$$h(t) = y_0 + \int_0^t v(s)ds\,.$$

We then have

$$\begin{aligned} |h(u_2) - h(u_1)| &\leq \int_{u_1}^{u_2} |v(s)|ds \\ &\leq \int_{u_1}^{u_2} |h_R(s)|ds\,. \end{aligned}$$

Set

$$U = \{y \in \Omega : \|y\|_\Omega < b + 1\}\,,$$

where b is defined in the proof of Theorem 3.1.

As a consequence of Step 2, Step 3 and (H4) together with the Ascoli-Arzela theorem we can conclude that $G_1 : \overline{U} \longrightarrow 2^\Omega$ is a compact multivalued map.

Step 4: G_1 has a closed graph.
Let $y_n \longrightarrow y_*,\ h_n \in G_1 y_n$ and $h_n \longrightarrow h_0$. We shall prove that $h_0 \in G_1 y_*$.
$h_n \in G_1(y_n)$ means that there exists $v_n \in S_{F_1,y_n}$ such that

$$h_n(t) = y_0 + \int_0^t v_n(s)ds\,.$$

We must prove that there exists $v_0 \in S_{F_1,y_*}$ such that

$$h_0(t) = y_0 + \int_0^t v_0(s)ds\,.$$

Consider the linear continuous operator $\Gamma : L^1(J,E) \longrightarrow C(J,E)$ defined by

$$(\Gamma v)(t) = \int_0^t v(s)ds\,.$$

We have

$$\|(h_n - y_0) - (h_0 - y_0)\|_\infty = \sup_{t\in J}[(h_n(t) - y_0) - (h_0(t) - y_0)] \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty\,.$$

From Lemma 2.1, it follows that $\Gamma \circ S_{F_1}$ is a closed graph operator.
Also from the definition of Γ we have that

$$h_n(t) - y_0 \in \Gamma(S_{F_1,y_n})\,.$$

This, besides to $y_n \longrightarrow y_*$ and Lemma 2.1, furnishs

$$h_0(t) = y_0 + \int_0^t v_0(s)ds$$

for some $v_0 \in S_{F_1,y_*}$.

Now, we show that $G_2 : \overline{U} \longrightarrow \Omega$ is a nonlinear contraction.
For this, let $y_1, y_2 \in \overline{U}$ In view of (H5) we get

$$\begin{aligned}
|(G_2y_1)(t) - (G_2y_2)(t)| &\leq \int_0^t |f_1(s, y_1(s)) - f_1(s, y_2(s))|ds \\
&\quad + \sum_{0<t_k<t} |I_k(y_1(t_k)) - I_k(y_2(t_k))| \\
&\leq \|q\|_{L^1}\phi(\|y_1 - y_2\|_\Omega) + \sum_{0<t_k<t} a_k|y_1(t_k) - y_2(t_k)| \\
&\leq \|q\|_{L^1}\phi(\|y_1 - y_2\|_\Omega) + \sum_{0<t_k<t} a_k\|y_1 - y_2\|_\Omega \\
&\leq \|q\|_{L^1}\phi(\|y_1 - y_2\|_\Omega) + \Big(\sum_{k=1}^{k=m} a_k\Big)\|y_1 - y_2\|_\Omega\,.
\end{aligned}$$

Set

$$\overline{\phi}(y) = \|q\|_{L^1}\phi(|y|) + \Big(\sum_{k=1}^{k=m} a_k\Big)|y|\,.$$

Clearly $\overline{\phi} : [0, \infty) \longrightarrow [0, \infty)$ is a continuous nondecreasing function with $\overline{\phi}(y) < y$ for $y > 0$.Thus by (H5) we obtain that G_2 is a nonlinear contraction.

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda Gy$ for any $\lambda \in (0, 1)$. As a consequence of Lemma 2.2 we deduce that G has a fixed point $y \in \overline{U}$ which is a solution of (NP). □

Acknowledgements

The authors are greatful to Professor J.J. Nieto for helpful discussions. Also the authors wish to thank the referee for interesting comments and remarks that lead to the improvement of the paper.

A. Boucherif wishes to thank KFUPM for its constant support.

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