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JET MANIFOLD ASSOCIATED TO A WEIL BUNDLE

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Abstract. Given a Weil algebra A and a smooth manifold M, we prove that the set J^AM of kernels of regular A-points of M, $\check{M}^A$, has a differentiable manifold structure and $\check{M}^A \longrightarrow J^AM$ is a principal fiber bundle.

It is well known that given a Weil algebra A, one can define a functor which associates to each smooth manifold M a manifold M^A whose elements are the points of M with values in A (see [2, 4]). When $A = \mathbb{R}^k_m$ (polynomials of degree $\leq k$ with m undetermined) it has been proved in [3] that the quotient manifold $J^r_m M$ under the action of the group $Aut(\mathbb{R}^k_m)$ exists. In this paper we show that this is still true for any A. This result was conjectured by I. Kolář [1]. The proof given here is based on the ideas of J. Muñoz-Díaz on this subject.

1. Preliminaries

Let A be a Weil algebra (finite dimensional local rational $\mathbb{R}$-algebra), $\mathfrak{m}_A$ its maximal ideal, $m = dim(\mathfrak{m}_A/\mathfrak{m}_A^2)$ and $\mathfrak{m}_A^{k+1} = 0$, $\mathfrak{m}_A^k \neq 0$. If the classes of $a_1, \dots, a_m$ generate $\mathfrak{m}_A/\mathfrak{m}_A^2$, one easily deduces that each element in A can be obtained as a polynomial on $a_1, \dots, a_m$.

For a given integer n, we define $\mathbb{R}^k_n \stackrel{def}{=} \mathbb{R}[\epsilon_1, \dots, \epsilon_n]/\mathfrak{m}^{k+1}$ where the ϵ's are undetermined variables and $\mathfrak{m}$ is the maximal ideal they generate.

We will denote by G the group of $\mathbb{R}$-algebra automorphisms of $\mathbb{R}^k_n$; that is, $G = Aut(\mathbb{R}^k_n)$. Note that G is a closed subgroup of the Lie group $GL(\mathfrak{m}/\mathfrak{m}^{k+1})$ (the linear automorphisms of the vector space $\mathfrak{m}/\mathfrak{m}^{k+1}$).

From now on, $\alpha\colon \mathbb{R}^k_n \longrightarrow A$ stands for a surjective $\mathbb{R}$-algebra morphism (hence $n \geq m$). Let $x_1, \dots, x_m$ be elements in $\mathfrak{m}$ such that $a_i = \alpha(x_i)$ (the a's as above); then, the classes of $x_1, \dots, x_m$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent and we can extend this collection to a basis $x_1, \dots, x_n$ with $\alpha(x_{m+j}) = 0$, $1 \leq j \leq n-m$: indeed, if (the classes of) $x_1, \dots, x_m, x'_{m+1}, \dots, x'_n$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$, and $\alpha(x'_{m+j}) = P_{m+j}(a_1, \dots, a_m)$, for polynomials P_{m+j}; then,

$$x_{m+j} \stackrel{def}{=} x'_{m+j} - P_{m+j}(x_1, \dots, x_m)$$

verify the required property.

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Lemma 1.1. *Let $\alpha, \beta \colon \mathbb{R}^k_n \longrightarrow A$ be $\mathbb{R}$-algebra epimorphisms; then there exists an automorphism $h \in G$ such that $\alpha = \beta \circ h$.*

Proof. It is sufficient to choose bases of $\mathfrak{m}/\mathfrak{m}^2$ as above with respect to α and β, respectively, and then to define h mapping the first to the second one. □

The subgroup of G of automorphisms of $\mathbb{R}^k_n$ which induce, by means of α, automorphisms of A is

$$\begin{aligned}\overline{G} &\stackrel{def}{=} \{g \in G \ / \ Ker(\alpha \circ g) = Ker(\alpha)\} \\ &= \{g \in G \ / \ g^{-1} Ker(\alpha) = Ker(\alpha)\}.\end{aligned}$$

This is a closed subgroup of G, so a manifold.

In this way, we have a morphism $\overline{G} \longrightarrow Aut(A)$ which is surjective by Lemma 1.1 and whose kernel is the closed subgroup

$$\overline{\overline{G}} \stackrel{def}{=} \{g \in G \ / \ \alpha \circ g = \alpha\}\,.$$

Hence,

Lemma 1.2. *There exists an isomorphism $\overline{G}/\overline{\overline{G}} \simeq Aut(A)$.*

2. Weil bundles

Let M be a smooth manifold of dimension n and B a Weil algebra.

Definition 2.1. The set M^B of the $\mathbb{R}$-algebra morphisms

$$p^B : \mathcal{C}^\infty(M) \longrightarrow B$$

is the so-called space of B-points of M or the Weil bundle of M associated to B. We will say that a B-point p^B is regular if it is surjective; the set of regular B-points of M will be denoted by $\check{M}^B$.

To simplify notation, when $B = \mathbb{R}^k_n$ we will write $\check{M}^k_n$ instead of $\check{M}^{\mathbb{R}^k_n}$.

Each function $f \in \mathcal{C}^\infty(M)$ defines a map $f^B : M^B \longrightarrow B$ by the rule $f^B(p^B) \stackrel{def}{=} p^B(f)$. There exists a differentiable structure on M^B determined by the condition that the maps f^B are smooth; now, $\check{M}^B$ is an open set of M^B. As examples, we have $M^{\mathbb{R}} = M$ and $M^1_1 = TM$, (see [2, 3]).

On the other hand, the tangent space $T_{p^B}M^B$ is canonically isomorphic to $Der_{\mathbb{R}}(\mathcal{C}^\infty(M), B)$, where each $X \in T_{p^B}M^B$ is related to the derivation $X' \in Der_{\mathbb{R}}(\mathcal{C}^\infty(M), B)$ determined by $X'(f) = X(f^B) \in B$, $f \in \mathcal{C}^\infty(M)$, where X derive each component of the vector function f^B (see [3]).

Lemma 2.2. *The following assertions holds.*

1. *$\check{M}^k_n \longrightarrow M$ is a principal fiber bundle with structure group G.*
2. *$\check{M}^A \longrightarrow M$ is a fiber bundle with typical fiber $G/\overline{\overline{G}}$.*
3. *$\alpha \colon \mathbb{R}^k_n \longrightarrow A$ induces a fiber bundle projection $\alpha_1 \colon \check{M}^k_n \longrightarrow \check{M}^A$ by the rule $\alpha_1(p^k_n) = \alpha \circ p^k_n$, $p^k_n \in \check{M}^k_n$.*

Proof. Almost everything in the claim is proved in [2]; anyway, for 2) and 3) we will construct local trivialization of $\check{M}^k_n$ via $\overline{G}$ and $\overline{\overline{G}}$.

Let $U \subset M$ be an open set trivializing $\check{M}^k_n \longrightarrow M$, in such a way that we have a local section, say $s_U : U \longrightarrow \check{U}^k_n$ and also a difeomorphism

$$\check{U}^k_n \xrightarrow{\sim} U \times G; \quad p^k_n \longrightarrow (p, g)$$

where $g \in G$ is the only automorphism of $\mathbb{R}^k_n$ such that $p^k_n = g \circ s_U(p)$.

On the other hand, for each $p^A \in \check{U}^A$ there exits a regular $\mathbb{R}^k_n$-point p^k_n such that $p^A = \alpha \circ p^k_n$; let $g \in G$ be the only automorphism of $\mathbb{R}^k_n$ with $p^k_n = g \circ s_U(p)$; then, $p^A = \alpha \circ g \circ s_U(p)$. Conversely, for each $g \in G$, the composition $\alpha \circ g \circ s_U(p)$ is a regular A-point in $\check{U}^A$. A necessary and sufficient condition to have $\alpha \circ g_1 \circ s_U(p) = \alpha \circ g_2 \circ s_U(p)$, $g_1, g_2 \in G$ is $g_1 g_2^{-1} \in \overline{\overline{G}}$. This way, we have a diffeomorphism

$$\check{U}^A \xrightarrow{\sim} U \times G/\overline{\overline{G}}; \quad p^A \longrightarrow (p, [g]_{\overline{\overline{G}}})$$

where $p^A = \alpha \circ g \circ s_U(p)$ and $[g]_{\overline{\overline{G}}}$ is the class of g in $G/\overline{\overline{G}}$.

Finally, by means of the above trivializations, the map $\alpha_1(p^k_n) = \alpha \circ p^k_n$ becomes locally the factor map by $\overline{\overline{G}}$:

$$U \times G \simeq \check{U}^k_n \xrightarrow{\alpha_1} \check{U}^A \simeq U \times G/\overline{\overline{G}}; \quad (p, g) \longrightarrow (p, [g]_{\overline{\overline{G}}}) . \qquad \square$$

3. A-JET MANIFOLD

Definition 3.1. The kernel of a regular A-point p^A will be called jet of p^A and we will denote it by $\mathfrak{p}^A = Ker(p^A)$. The set of jets of regular A-points will be called the space of A-jets and denoted by $J^A M$; thus, we have a surjective map $Ker \colon \check{M}^A \longrightarrow J^A M$ which associates to each A-point its kernel.

In what follows we endow $J^A M$ with a smooth structure.

Using local trivializations of $\check{M}^A$ (see the above section), the map Ker may be written as

$$U \times G/\overline{\overline{G}} \simeq \check{U}^A \xrightarrow{Ker} J^A M; \quad (p, [g]_{\overline{\overline{G}}}) \longrightarrow Ker(\alpha \circ g \circ s_U(p)) .$$

Two couples $(p, [g_1]_{\overline{\overline{G}}})$ and $(p, [g_2]_{\overline{\overline{G}}})$ have the same image by Ker if and only if $g_1 g_2^{-1} \in \overline{G}$; so, we can think of Ker as a factor map by $\overline{G}/\overline{\overline{G}}$. Bearing in mind this idea, we make the following construction. For each trivializing open set U as above, we define a bijective map

$$U \times G/\overline{G} \xrightarrow{\phi_U} J^A U; \quad (p, [g]_{\overline{G}}) \longrightarrow Ker(\alpha \circ g \circ s_U(p)) .$$

The family $(J^A U, \phi_U)_U$ is an atlas in a general sense; in fact, if U, V are open sets as above and $h_p \in G$ is the only automorphism such that $s_U(p) = h_p \circ s_V(p)$, then the transition functions

$$\phi_{UV} \colon (U \cap V) \times G/\overline{G} \xrightarrow{\phi_U^{-1}} J^A(U \cap V) \xrightarrow{\phi_V} (U \cap V) \times G/\overline{G}$$

are defined as $\phi_{UV}(p, [g]_{\overline{G}}) = (p, [g h_p]_{\overline{G}})$, therefore they are smooth.

Remark 3.2. This differentiable structure is independent of the choice of α; indeed, if $\beta \colon \mathbb{R}^k_n \longrightarrow A$ is another surjective morphism and we carry on the same construction, the transition between the corresponding charts is realized by means of an $h \in G$ with $\alpha = \beta \circ h$ (see Lemma 1.1).

Theorem 3.3. *On $J^A M$ there exists a differentiable structure such that*

$$Ker \colon \check{M}^A \longrightarrow J^A M$$

is a principal fiber bundle with group $Aut(A)$.

Proof. Fixing α, with the above notation, the map Ker becomes locally the factor map by the group $\overline{G}/\overline{\overline{G}}$:

$$U \times G/\overline{\overline{G}} \simeq \check{U}^A \xrightarrow{Ker} J^A U \simeq U \times G/\overline{G}\,.$$

We finish the proof by taking into account that the local action of $\overline{G}/\overline{\overline{G}}$ corresponds with the global action of $Aut(A)$ on $\check{M}^A$ given by $\sigma \cdot p^A \overset{def}{=} \sigma \circ p^A$, $\sigma \in Aut(A)$ and $p^A \in \check{M}^A$ (see also Lemma 1.2). □

The tangent space of $\check{M}^A$ at p^A projects onto that of $J^A M$ at $\mathfrak{p}^A$; therefore, $T_{\mathfrak{p}^A} J^A$ is a quotient space of $T_{p^A} J^A \simeq Der_{\mathbb{R}}(\mathcal{C}^\infty(M), A)$ (see Section 2). By definition, the vector functions f^A, $f \in \mathfrak{p}^A$, vanish on the fiber $(Ker)^{-1}(\mathfrak{p}^A)$; it follows that the vertical tangent space $T^v_{p^A} J^A M \subset Der_{\mathbb{R}}(\mathcal{C}^\infty(M), A)$ kills $\mathfrak{p}^A$ thus that space can be identified with a subset of $Der_{\mathbb{R}}(\mathcal{C}^\infty(M)/\mathfrak{p}^A, A)$; but, the Lie algebra of $Aut(A)$ is $Der_{\mathbb{R}}(A, A)$ and this clearly forces $T^v_{p^A} J^A \simeq Der_{\mathbb{R}}(\mathcal{C}^\infty(M)/\mathfrak{p}^A, A)$. Finally, by using p^A we have $\mathcal{C}^\infty(M)/\mathfrak{p}^A \simeq A$ and

Proposition 3.4. *For each $\mathfrak{p}^A \in J^A M$, the following isomorphism holds,*

$$T_{\mathfrak{p}^A} J^A M \simeq Der_{\mathbb{R}}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)/\mathfrak{p}^A)/Der_{\mathbb{R}}(\mathcal{C}^\infty(M)/\mathfrak{p}^A, \mathcal{C}^\infty(M)/\mathfrak{p}^A)\,.$$

4. Immersion into a Grassmannian

If $\mathfrak{p}^A \in J^A M$ projects onto $p \in M$, then $\mathfrak{m}_p^{k+1} \subseteq \mathfrak{p}^A \subseteq \mathfrak{m}_p$. This way, $\mathfrak{p}^A$ is identified with a linear subspace of $\mathfrak{m}_p/\mathfrak{m}_p^{k+1}$ with dimension $d = dim(Ker(\alpha))$. If $Gr(d, \mathfrak{m}_p/\mathfrak{m}_p^{k+1})$ denotes the Grassmann manifold of d-planes of $\mathfrak{m}_p/\mathfrak{m}_p^{k+1}$, we have an inclusion $(J^A M)_p \subseteq Gr(d, \mathfrak{m}_p/\mathfrak{m}_p^{k+1})$, where $(J^A M)_p$ is the fiber of $J^A M$ at p.

Let $T^{*,k}M$ be the k-th cotangent fiber bundle of M (so, $(T^{*,k}M)_p = \mathfrak{m}_p/\mathfrak{m}_p^{k+1}$) and $Gr(d, T^{*,k}M)$ the corresponding Grassmann manifold of d-planes. If U is an open set as in the above sections, we get an isomorphism $\mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1} \overset{s_U(p)}{\simeq} \mathbb{R}_n^k$ for each $p \in U$, hence, $s_U(p)^{-1} Ker(\alpha)$ is a linear d-dimensional subspace of $\mathfrak{m}_p/\mathfrak{m}_p^{k+1}$. One deduces the following trivialization of $Gr(d, T^{*,k}M)$,

$$U \times GL(\mathfrak{m}/\mathfrak{m}^{k+1})/\overline{GL}(\mathfrak{m}/\mathfrak{m}^{k+1}) \xrightarrow{\sim} Gr(d, T^{*,k}U)\,;$$

$$(p, g) \longrightarrow s_U(p)^{-1} \circ g^{-1}(Ker(\alpha)) = s_U(p)^{-1} Ker(\alpha \circ g)\,,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathbb{R}_n^k$, $GL(\mathfrak{m}/\mathfrak{m}^{k+1})$ the linear group of $\mathfrak{m}/\mathfrak{m}^{k+1}$ and $\overline{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ the isotropy group of $Ker(\alpha)$, that is,

$$\{g \in GL(\mathfrak{m}/\mathfrak{m}^{k+1}) \;/\; Ker(\alpha \circ g) = Ker(\alpha)\}\,.$$

Theorem 4.1. *The space of A-jets, $J^A M$, is a submanifold of the grassmannian $Gr(d, T^{*,k}M)$; moreover, that inclusion is a morphism of fibered bundles on M.*

Proof. Recall that $G = Aut(\mathbb{R}_n^k)$ is a closed subgroup of $GL(\mathfrak{m}/\mathfrak{m}^{k+1})$; for each coordinate open set U, the inclusion of $J^A M$ into $Gr(d, T^{*,k}M)$ becomes the map

$$U \times G/\overline{G} \longrightarrow U \times GL(\mathfrak{m}/\mathfrak{m}^{k+1})/\overline{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$$

induced by the immersion of G into $GL(\mathfrak{m}/\mathfrak{m}^{k+1})$. This finishes the proof. □

Remark 4.2. Note that we have shown implicitly that $(J^A M)_p$ is identified with the orbit through $\mathfrak{p}^A \in Gr(d, T^{*,k}M)_p$ by the action of the group $Aut(\mathcal{C}^\infty(M)/\mathfrak{m}_p^{k+1})$.

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