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A NOTE ON SOME DISCRETE VALUATION RINGS OF ARITHMETICAL FUNCTIONS

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Abstract. The paper studies the structure of the ring A of arithmetical functions, where the multiplication is defined as the Dirichlet convolution. It is proven that A itself is not a discrete valuation ring, but a certain extension of it is constructed,this extension being a discrete valuation ring. Finally, the metric structure of the ring A is examined.

1. Introduction

In [6], K. L. Yokom investigated the prime factorization of arithmetical functions (mappings from $\mathbf{N}^*$ into $\mathbf{C}$) in a certain subring of the regular convolution ring. In the unitary ring $(A, +, *_\mho)$ of the arithmetical functions, where the unitary convolution $*_\mho$ of two arithmetical functions $f,\ g \in A$ is defined by:

$$(f *_\mho g)(n) = \sum_{d|n,(d,\frac{n}{d})=1} f(d)g(\frac{n}{d}), \tag{1}$$

K. L. Yokom considered the subring $B_\mho$:

$$B_\mho = \{f \in A | \omega(m) = \omega(n) \text{ implies } f(m) = f(n)\}, \tag{2}$$

where $\omega(m)$ is the number of distinct prime divisors of m and proved the following:

Theorem 1.1. ([6]) *The ring $B_\mho$ contains only one prime π (up to associates) and each nonzero $f \in B_\mho$ can be written uniquely in the form*

$$f = u *_\mho \pi^{\omega(N(f))},$$

where u is a unit in $B_\mho$ and $N(f)$ is given by:

$$N(f) = min\{n | f(n) \neq 0\}.$$

We observe that $\eta : \mathbf{C}[[X]] \to B_\mho$ defined as

$$\eta(\sum_{k=0}^{\infty} a_k X^k)(n) = \omega(n)! a_{\omega(n)} \tag{3}$$

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is a ring-isomorphism (see [5]) and therefore $(B_{\mho}, +, *_{\mho})$ is a discrete valuation ring. This proves Yokom's Theorem. (It is clear that $\pi = \eta(X)$ and therefore $\pi(n) = 1$ if n is a prime power $p^{\alpha} > 1$ and $\pi(n) = 0$ otherwise.)

In the lattice of the regular convolutions, the unitary convolution is the zero element, and the Dirichlet convolution is the universal element (see [2]). The Dirichlet convolution $*_D$ of two arithmetical functions $f,\ g \in A$ is defined by:

$$(f *_D g)(n) = \sum_{d|n} f(d) g(\frac{n}{d}) . \tag{4}$$

K. L. Yokom determined a discrete valuation subring of the unitary ring of arithmetical functions $(A, +, *_{\mho})$. Our purpose is to find a discrete valuation ring which is an extension of the ring $(A, +, *_D)$.

2. Main results

First we will try to get some properties of the ring $(A, +, *_D)$.
It is well known that it is a local ring, his maximal ideal being
$M = A \setminus U(A) = \{f \in A | f(1) \neq 0\}$. Unlike $(A, +, *_{\mho})$, the ring $(A, +, *_D)$ is an integrity domain.

Let $p_1 < p_2 < \ldots$ be the set of the primes. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ is a nonzero natural number, let $\Omega(n)$ be the total number of prime factors of n, each being counted according to its multiplicity, that is

$$\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r .$$

Ω is obviously a monoid-morphism between $(\mathbf{N}^*, \cdot)$ and $(\mathbf{N}, +)$.
For every $k \in \mathbf{N}$ we put

$$I_k = \{f \in A | f(n) = 0 \text{ for every } n \in \mathbf{N}^* \text{ such that } (n, p_1 p_2 \ldots p_k) = 1\}$$

and

$$J_k = \{f \in A | f(n) = 0 \text{ for every } n \in \mathbf{N}^* \text{ such that } \Omega(n) < k\} .$$

Proposition 2.1. a) *I_k and J_k are ideals in $(A, +, *_D)$ for every $k \in \mathbf{N}$.*

b) $\{0\} = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_k \subset I_{k+1} \subset \ldots ,\quad \bigcup_{k \geq 0} I_k = A.$

c) $A = J_0 \supset M = J_1 \supset J_2 \supset \ldots \supset J_k \supset J_{k+1} \supset \ldots ,\quad \bigcap_{k \geq 0} J_k = \{0\}.$
In particular, the ring $(A, +, *_D)$ is neither noetherian, nor artinian.

Proof. a) Let $f, g \in I_k,\ h \in A$, and let $n \in \mathbf{N}^*$ such that $(n, p_1 p_2 \ldots p_k) = 1$. Then for every divisor d of n we have $(d, p_1 p_2 \ldots p_k) = 1$ and therefore

$$(f - g)(n) = f(n) - g(n) = 0 ,$$
$$(f *_D h)(n) = \sum_{d|n} f(d) h(\frac{n}{d}) = 0 .$$

Now let $f, g \in J_k,\ h \in A$, and let $n \in \mathbf{N}^*$ such that $\Omega(n) < k$. For every divisor d of n we have $\Omega(d) \leq \Omega(n) < k$ and therefore

$$(f - g)(n) = f(n) - g(n) = 0,$$

$$(f *_D h)(n) = \sum_{d|n} f(d)h(\frac{n}{d}) = 0\,.$$

b) and c) are obvious. □

An interesting property of the ideals J_k is the following one.

Proposition 2.2. *Let k, l be natural numbers and f, g be arithmetical functions, $f \in J_k \setminus J_{k+1}$, $g \in J_l \setminus J_{l+1}$. Then $f *_D g \in J_{k+l} \setminus J_{k+l+1}$.*

Proof. At the beginning we will prove that $f, g \in J_{k+l}$.
Let $n \in \mathbf{N}^*$ such that $\Omega(n) < k + l$. If d is a divisor of n, then $\Omega(d) < k$ or $\Omega(\frac{n}{d}) < l$. It results

$$(f *_D g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = 0, \text{ that is } f *_D g \in J_{k+l}\,.$$

It remains to prove that there exists $n \in \mathbf{N}^*$ such that $\Omega(n) = k + l$ and $(f *_D g)(n) \neq 0$.
If $l = 0$, then $g(1) \neq 0$ and we can find $n \in \mathbf{N}^*$ with

$$\Omega(n) = k,\ f(n) \neq 0,\ f(d) = 0\ \forall d \in \mathbf{N}^* \setminus \{n\}, d|n\,.$$

We get

$$(f *_D g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = f(n)g(1) \neq 0\,.$$

The assertion can be proved similarly if $k = 0$. Therefore one may assume that $k, l \neq 0$.

From the hypothesis $f \notin J_{k+1}$ we obtain a natural number m with $\Omega(m) = k$ and $f(m) \neq 0$. Let $m = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$ be the decomposition of m into prime factors, where $q_1, q_2, \ldots, q_s$ are mutually distinct, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_s > 0$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_s = k$. We may choose m in the set
$\mathcal{M} = \{m \in \mathbf{N}^* | \Omega(m) = k, f(m) \neq 0\}$ so that the vector

$$(\alpha_1, \alpha_2, \ldots, \alpha_s, \underbrace{0, \ldots, 0}_{k-s})$$

is maximal in the lexicographycal ordering. We keep fixed such a number m and also the corresponding values $s, q_1, \ldots, q_s, \alpha_1, \ldots, \alpha_s$.
Similarly, there exists $n \in \mathbf{N}^*$ with $\Omega(n) = l$ and $g(n) \neq 0$. Let $n = q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s} r_1^{x_1} r_2^{x_2} \ldots r_t^{x_t}$ the decomposition of n into prime factors, where $t \geq 0$, $r_1, r_2, \ldots, r_t$ are mutually distinct primes, $\{q_1, \ldots, q_s\} \cap \{r_1, \ldots, r_t\} = \emptyset$, $\beta_1, \ldots, \beta_s$, $x_1, \ldots, x_t \in \mathbf{N}$, $x_1 \geq x_2 \geq \ldots \geq x_t > 0$ and $\beta_1 + \ldots + \beta_s + x_1 + \ldots + x_t = l$. We may choose n in the set $\mathcal{N} = \{n \in \mathbf{N}^* | \Omega(n) = l, g(n) \neq 0\}$ so that $\beta_1 + \ldots + \beta_s$ is maximal and also the vector $(\alpha_1 + \beta_1, \ldots, \alpha_s + \beta_s)$ is maximal in the lexicographycal ordering. We keep fixed such a number n and also the corresponding values $\beta_1, \ldots, \beta_s, t, r_1, \ldots, r_t, x_1, \ldots, x_t$.

Let now $d \in \mathbf{N}^*$, $d|mn$, with the property $f(d)g(\frac{mn}{d}) \neq 0$. From the relations

$$\Omega(d) \geq k,\ \Omega(\frac{mn}{d}) \geq l,\ \Omega(mn) = k + l$$

we get $\Omega(d) = k$ and $\Omega(\frac{mn}{d}) = l$. Hence

$$d = q_1^{\gamma_1} \dots q_s^{\gamma_s} r_1^{y_1} \dots r_t^{y_t} \text{ and } \frac{mn}{d} = q_1^{\alpha_1+\beta_1-\gamma_1} \dots q_s^{\alpha_s+\beta_s-\gamma_s} r_1^{x_1-y_1} \dots r_t^{x_t-y_t},$$

where

$$\gamma_1, \dots, \gamma_s, y_1, \dots, y_t \in \mathbf{N}, \quad \gamma_i \leq \alpha_i + \beta_i \ \forall i \in \{1, 2, \dots, s\},$$

$$y_j \leq x_j \ \forall j \in \{1, 2, \dots, t\}, \quad \sum_{i=1}^{s} \gamma_i + \sum_{j=1}^{t} y_j = k = \sum_{i=1}^{s} \alpha_i .$$

We observe that $\frac{mn}{d} \in \mathcal{N}$. Because the way we have chosen n, it results succesively

$$\beta_1 + \dots + \beta_s \geq (\alpha_1 + \beta_1 - \gamma_1) + \dots + (\alpha_s + \beta_s - \gamma_s),$$

$$\sum_{i=1}^{s} \gamma_i \geq \sum_{i=1}^{s} \alpha_i = \sum_{i=1}^{s} \gamma_i + \sum_{j=1}^{t} y_j ,$$

$$y_1 = y_2 = \dots = y_t = 0 .$$

Moreover, from the maximality of $(\alpha_1 + \beta_1, \dots, \alpha_s + \beta_s)$ we get

$$(\alpha_1 + \beta_1, \dots, \alpha_s + \beta_s) \geq (\alpha_1 + (\alpha_1 + \beta_1 - \gamma_1), \dots, \alpha_s + (\alpha_s + \beta_s - \gamma_s)),$$

and therefore $\gamma_1 \geq \alpha_1$. If $(\gamma_{i_1}, \dots, \gamma_{i_s})$ is a permutation of the numbers $(\gamma_1, \dots, \gamma_s)$ realized in such a way that $\gamma_{i_1} \geq \dots \geq \gamma_{i_s}$, then

$$d = q_{i_1}^{\gamma_{i_1}} \dots q_{i_s}^{\gamma_{i_s}} \in \mathcal{M} .$$

In accordance with the choosing of m one may write

$$(\alpha_1, \dots, \alpha_s, \underbrace{0, \dots, 0}_{k-s}) \geq (\gamma_{i_1}, \dots, \gamma_{i_s}, \underbrace{0, \dots, 0}_{k-s}) .$$

We obtain $\gamma_{i_1} = \gamma_1 = \alpha_1$ and, by induction, $\gamma_i = \alpha_i$ for every $i \in \{1, 2, \dots, s\}$.

In conclusion,

$$d = m, \quad \frac{mn}{d} = n, \quad (f *_D g)(mn) = f(m)g(n) \neq 0 ,$$

and therefore $f *_D g \notin J_{k+l+1}$. □

Now we can define the degree $D(f)$ of a (nonzero) arithmetical function as follows:

$$D(f) = \max\{k \in \mathbf{N} | f \in J_k\} . \tag{5}$$

Obviously, $D(f) = 0 \Leftrightarrow f \in U(A)$.

Proposition 2.3. *The degree $D : A \setminus \{0\} \to \mathbf{N}$ has the following properties:*

i) *D is a surjective mapping.*

ii) *$D(f *_D g) = D(f) + D(g)$ $\forall f, g \in A \setminus \{0\}$.*

iii) *$D(f + g) \geq min(D(f), D(g))$ $\forall f, g \in A \setminus \{0\}$, $g \neq -f$.*

Proof. i) Let $k \in \mathbf{N}$. The function $f : \mathbf{N}^* \to \mathbf{C}$

$$f(n) = \begin{cases} 1 & \text{if} \quad n = 2^k \\ 0 & \text{if} \quad n \in \mathbf{N}^* \setminus \{2^k\} \end{cases}$$

verifies $D(f) = k$.

ii) Is a direct consequence of Proposition 2.2.

iii) Let $k = D(f)$, $l = D(g)$. One may assume that $k \geq l$. Then $f \in J_k \subseteq J_l$, $g \in J_l$, hence $f + g \in J_l$. We derive that $D(f + g) \geq l$. $\square$

Now we can extend the degree mapping D to the field of fractions $K = \{\frac{f}{g} | f, g \in A, g \neq 0\}$ of A, by putting

$$\bar{D} : K \setminus \{0\} \to \mathbf{Z} \quad \bar{D}(\frac{f}{g}) = D(f) - D(g) \ \forall f, g \in A \setminus \{0\}\,. \tag{6}$$

$\bar{D}$ is obviously well-defined.

Proposition 2.4. *$\bar{D}$ has the following properties:*
i) *$\bar{D}$ is surjective.*

ii) $\bar{D}(x *_D y) = \bar{D}(x) + \bar{D}(y) \ \forall x, y \in K \setminus \{0\}$.

iii) $\bar{D}(x + y) \geq min(\bar{D}(x), \bar{D}(y)) \ \forall x, y \in K \setminus \{0\}, \ y \neq -x$.

Proof. The first two statements follow imediately from Proposition 2.3.
iii) If $x = \frac{f_1}{g_1}, \ y = \frac{f_2}{g_2}, \ f_1, f_2, g_1, g_2 \in K \setminus \{0\}$, then

$$\begin{aligned} \bar{D}(x+y) = \bar{D}\left(\frac{f_1 *_D g_2 + f_2 *_D g_1}{g_1 *_D g_2}\right) &= D(f_1 *_D g_2 + f_2 *_D g_1) - D(g_1 *_D g_2) \\ &\geq \min(D(f_1 *_D g_2), D(f_2 *_D g_1)) - D(g_1) - D(g_2) \\ &= \min(D(f_1) + D(g_2), D(f_2) + D(g_1)) - D(g_1) - D(g_2) \\ &= \min(D(f_1) - D(g_1), D(f_2) - D(g_2)) \\ &= \min\left(\bar{D}\left(\frac{f_1}{g_1}\right), \bar{D}\left(\frac{f_2}{g_2}\right)\right) = \min(\bar{D}(x), \bar{D}(y))\,. \end{aligned}$$

$\square$

For any $a \in (1, +\infty)$ one defines $v : K \to \mathbf{R}$

$$v(x) = \begin{cases} a^{-\bar{D}(x)} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0\,. \end{cases}$$

Theorem 2.1. i) *v is a non-archimedean valuation on K.*

ii) *$B_D = \{\frac{f}{g} \in K | v(\frac{f}{g}) \leq 1\}$ is a discrete valuation ring and A is canonically embedded in B_D.*

iii) *$P_D = \{\frac{f}{g} \in K | v(\frac{f}{g}) < 1\}$ is the unique nontrivial prime ideal of B_D.*

Proof. The first assertion follows from Proposition 3.1.10 of [1], and the other two assertions are contained in Proposition 3.1.16 of [1]. □

Remark 2.1. *$(A, +, *_D)$ is not a discrete valuation ring, because the ideals $\{f \in A | f(1) = f(2) = 0\}$ and $\{f \in A | f(1) = f(3) = 0\}$ are not comparable.*

If $\delta_m : \mathbf{N}^* \to \mathbf{C}$ $(m \in \mathbf{N}^*)$ is the arithmetical function defined by

$$\delta_m(n) = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \,, \end{cases}$$

then we get the following obvious results:

Corollary 2.1. *The ring B_D contains only one nonzero prime, $\frac{\delta_2}{\delta_1}$ (up to associates), and each nonzero element $x \in B_D$ may be written uniquely in the form*

$$x = u *_D \left(\frac{\delta_2}{\delta_1}\right)^{\bar{D}(x)} ,$$

where $u \in U(B_D) = \{x \in K | v(x) = 1\}$.

Corollary 2.2. *Let f and g be two nonzero arithmetical functions such that $D(f) \geq D(g)$. Then there are two arithmetical functions, h and k, with $D(h) = D(k)$ and*

$$f *_D k = g *_D h *_D \delta_{2^{D(f)-D(g)}} .$$

One can define on K a distance, putting

$$d(x, y) = v(x - y) \;\forall x, y \in K .$$

The restriction of d to the ring $(A, +, *_D)$ is also a distance, defined by

$$d(f, g) = \begin{cases} a^{-D(f-g)} & \text{if} \quad f \neq g \\ 0 & \text{if} \quad f = g . \end{cases}$$

The structure of the metric space (A, d) is established by

Theorem 2.2. *The metric space (A, d) is complete.*

Proof. Let $(f_n)_{n\geq 0}$ be a Cauchy sequence in A. Then for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbf{N}$ such that

$$a^{-D(f_m - f_n)} < \varepsilon \;\forall m, n \geq N_\varepsilon .$$

For each $k \in \mathbf{N}$, taking $\varepsilon = a^{-k}$ we get:
there exists $N_k \in \mathbf{N}$ such that

$$D(f_m - f_n) > k \;\forall m, n \in \mathbf{N},\; m, n \geq N_k ,$$

that is $f_m(r) = f_n(r)$ for every $r \in \mathbf{N}^*$ with $\Omega(r) \leq k$. Choosing for each $k \in \mathbf{N}$ the lowest natural number N_k with the property above, we have

$$N_0 \leq N_1 \leq \ldots \leq N_k \leq N_{k+1} \leq \ldots$$

One defines the function $f : \mathbf{N}^* \to \mathbf{C}$ by

$$f(r) = f_{N_{\Omega(r)}}(r) \;\forall r \in \mathbf{N}^*$$

and one proves that f is the limit of the sequence $(f_n)_{n\geq 0}$.

Let $\varepsilon > 0$, $k = \max([-\ln\varepsilon], 0)$ and $N_k \in \mathbf{N}$ defined as before. If $n \geq N_k$ and if $r \in \mathbf{N}^*$ with $\Omega(r) \leq k$, then $N_{\Omega(r)} \leq N_k \leq n$. It follows

$$f_n(r) = f_{N_{\Omega(r)}}(r) = f(r)\,,$$

hence $D(f_n - f) > k$, and therefore $d(f_n, f) < \varepsilon$.

Consequently, $\lim_{n\to\infty} f_n = f$ and the Theorem is proved. □

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