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BOUNDED SOLUTIONS AND ASYMPTOTIC STABILITY OF NONLINEAR DIFFERENCE EQUATIONS IN THE COMPLEX PLANE

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Abstract. An existence and uniqueness theorem for solutions in the Banach space l_1 of a nonlinear difference equation is given. The constructive character of the proof of the theorem predicts local asymptotic stability and gives information about the size of the region of attraction near equilibrium points.

1. Introduction

We consider the m-th order non-linear difference equation of the form

$$f(n+m)+\sum_{p=1}^{m}(\alpha_p+\beta_p(n))f(n+m-p)=\sum_{i=1}^{N}c_i(n)f(n+q_{i1})f(n+q_{i2})$$
$$+\sum_{j=1}^{M}d_j(n)f(n+q_{j3})f(n+q_{j4})f(n+q_{j5}) \tag{1.1}$$

where m,N,M are positive integers, q_{i1}, q_{i2}, $i=1,\dots,N$, q_{j3}, q_{j4}, q_{j5}, $j=1,\dots,M$ are non-negative integers, α_p, $p=1,\dots,m$ in general complex numbers, with the initial conditions

$$f(p)=u_p, p=1,\dots,m \tag{1.2}$$

Under suitable assumptions on the sequences $\beta_p(n)$,$p=1,\dots,m$, $c_i(n)$, $i=1,\dots,N$, $d_j(n)$, $j=1,\dots,M$ and the roots of the polynomial $r^m+\alpha_1 r^{m-1}+\dots+\alpha_m=0$, we prove that there exists a unique solution of (1.1), (1.2) in the Banach space l_1 of all bounded complex sequences $f(n)$ which satisfy the condition $\sum_{n=1}^{\infty}|f(n)|<+\infty$. For the motivation of seeking solutions of nonlinear difference

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equations in l_1 see [3]. More general nonlinearities concerning the right-hand side of equation (1.1) are considered in a forthcoming paper.

The method used is a functional analytic method based on a method which is developed by E. K. Ifantis in [3]. Using this method, equation (1.1) is reduced equivalently to an operator equation of the form

$$f = Q(f) + \xi_1 = \phi(f)\,, \tag{1.3}$$

where ξ_1 is a fixed element in an abstract Banach space H_1 depending on the initial conditions $f(p) = u_p, p = 1,\ldots,m$ and Q a non-linear operator which is defined on H_1. Under suitable assumptions on the sequences $c_i(n)$, $i = 1,\ldots,N$, $d_j(n)$, $j = 1,\ldots,M$, it is shown that the non-linear operator $\phi : B(0,R_0) \to B(0,R_0)$, where $B(0,R_0)$ is an open ball centered at the origin of H_1, is a holomorphic map in $B(0,R_0)$, i.e. its Fréchet derivative exists at every point in the open ball $B(0,R_0)$ in H_1 and $\phi(B(0,R_0) \subset B(0,R_0)$. For holomorphic maps the following result of Earle and Hamilton [2] holds:

If $f : X \to X$ is holomorphic and f(X) lies strictly inside X, then f has a unique fixed point in X, where X is a bounded, connected and open subset of a Banach space E.

By saying that a subset X' of X lies strictly inside X we mean that there exists a $\epsilon_1 > 0$ such that $\|x' - y\| > \epsilon_1$ for all $x' \in X'$ and $y \in E - X$.

Using the above it is proved that the equation (1.3) has a unique fixed point in the space H_1. This means equivalently that the initial value problem (1.1), (1.2) has a unique solution in the space l_1.

The above result establishes local asymptotic stability and gives information about the size of the region of attraction near equilibrium points. Moreover, we can see in particular cases how the size of the region of attraction depends on the parameters of equation (1.1) and the initial conditions (1.2). Also, it is of some interest to note that we can actually obtain in some cases an explicit upper bound for the solution $f(n)$ of equation (1.1) which satisfies the initial conditions (1.2).

In some cases we can also find the radius of convergence of the power series $f(z) = \sum_{n=1}^{\infty} f(n)z^{n-1}$. This power series corresponding to $f(n)$ is called generating function and it may be a formal solution of a differential or integral equation.

In Section 2, equation (1.1) together with the initial conditions (1.2) is reduced to an operator equation of the form (1.3). In Section 3 a theorem is proved for the existence and uniqueness of solutions in l_1 of the initial value problem (1.1), (1.2). The proof of the theorem is based on two lemmas. In the first lemma we give the abstract forms of the right-hand side (nonlinear) part of the difference equation (1.1) defined on H_1, and in the second lemma we prove that the nonlinear operators which are the abstract forms presented in the first lemma are Fréchet differentiable in H_1. Moreover we find the Fréchet derivative for each one of them. These two lemmas together with the results of Section 2 and the Earle-Hamilton fixed point theorem give Theorem 3.1. Finally in Section 4 we apply the theorem for some nonlinear difference equations which can be deduced from equation (1.1).

2. The abstract form of nonlinear difference equations

In the following H is used to denote an abstract separable Hilbert space with the orthonormal basis $e_n, n = 1, 2, 3, \ldots$. We use the symbols $(\cdot, \cdot)$ and $\|\cdot\|$ to denote scalar product and norm in H respectively. By H_1 we mean the Banach space consisting of those elements f in H which satisfy the condition $\sum_{n=1}^{\infty} |(f, e_n)| < +\infty$. The norm in H_1 is denoted by $\|f\|_1 = \sum_{n=1}^{\infty} |(f, e_n)|$. By $f(n)$ we mean an element of the Banach space l_1 and by $f = \sum_{n=1}^{\infty} f(n) e_n$ we mean that element in H_1 generated by $f(n) \in l_1$. The norm in l_1 is denoted by $\|f(n)\|_{l_1} = \sum_{n=1}^{\infty} |f(n)|$. Finally by V we mean the shift operator on H

$$V : V e_n = e_{n+1}\,, \quad n = 1, 2, \ldots$$

and by V^* its adjoint

$$V^* : V^* e_n = e_{n-1}\,, \quad n = 2, 3, \ldots, V^* e_1 = 0\,.$$

One can easily prove that the function

$$\phi : H_1 \to l_1$$

which is defined as follows:

$$\phi(f) = (f, e_n) = f(n)$$

is an isomorphism from H_1 onto l_1. We call f the abstract form of $f(n)$.

In general, if G is a mapping in l_1 and N is a mapping in H_1, we call $N(f)$ the abstract form of $G(f(n))$ if

$$G(f(n)) = (N(f), e_n) \tag{2.1}$$

It follows easily that $V^* f$ is the abstract form of $f(n+1)$, since

$$f(n+1) = (f, e_{n+1}) = (V^* f, e_n)\,,$$

Vf is the abstract form of $f(n-1)$, since

$$f(n-1) = (f, e_{n-1}) = (Vf, e_n)$$

and $B_p f$, $p = 1, 2, \ldots, m$ are the abstract forms of $\beta_p(n) f(n)$, where B_p are the diagonal operators

$$B_p e_n = \beta_p(n) e_n\,, \quad n = 1, 2, \ldots, m \tag{2.2}$$

since $\beta_p(n) f(n) = \beta_p(n)(f, e_n) = (f, \beta_p^*(n) e_n) = (B_p f, e_n)$.

It follows readily from the above that the abstract form of the left-hand side (linear) part of (1.1) is

$$V^{*m}f+\sum_{p=1}^{m}(\alpha_p+B_p)V^{*m-p}f$$

because

$$\begin{aligned} f(n+m)&+\sum_{p=1}^{m}(\alpha_p+\beta_p(n))f(n+m-p)\\ &=(V^{*m}f+\sum_{p=1}^{m}(\alpha_p+B_p)V^{*m-p}f,e_n)\,,\quad n=1,2,\dots \end{aligned}\tag{2.3}$$

From (2.1), taking into account (2.3), we obtain the abstract form of equation (1.1):

$$V^{*m}f+\sum_{p=1}^{m}(\alpha_p+B_p)V^{*m-p}f=\sum_{i=1}^{N}N_i(f)+\sum_{j=1}^{M}K_j(f)\,,$$

where the second part of this equation is the abstract form of the right-hand side (nonlinear) part of (1.1) and the nonlinear operators $N_i(f), 1\le i\le N$ and $K_j(f)$, $1\le j\le M$ are defined in H_1. Since $V^*V=I$ the above equation can be written as follows:

$$\begin{aligned} V^*(I-r_1V)(I-r_2V)\dots(I-r_mV)f&+\sum_{p=1}^{m}B_pV^{*m-p}f\\ &=\sum_{i=1}^{N}N_i(f)+\sum_{j=1}^{M}K_j(f) \end{aligned}\tag{2.4}$$

where r_p, $p=1,2,\dots,m$ are the roots of the equation:

$$r^m+\alpha_1r^{m-1}+\dots+\alpha_m=0\tag{2.5}$$

and I is the identity operator, or

$$V^{*m}\Gamma f=\sum_{i=1}^{N}N_i(f)+\sum_{j=1}^{M}K_j(f)\tag{2.6}$$

where

$$\Gamma=(I-r_1V)(I-r_2V)\dots(I-r_mV)+V^m\sum_{p=1}^{m}B_pV^{*m-p}f\,.\tag{2.7}$$

The operator Γ leaves invariant the space H_1, i.e. for every $x\in H_1$, $\Gamma x\in H_1$ and therefore equation (2.4) can be considered as an equation in H_1 whenever Γ is defined from H_1 into H_1 [3]. This means that f is a solution of (2.4) or (2.6) in H_1 if and only if $\{f(n)\}$ is a solution of (1.1) in l_1. Also the operator Γ has a bounded inverse on H_1 provided that $|r_p|<1$, $p=1,2,\dots,m$ [3].

Taking into account the above properties of the operator Γ it can be proved similarly as in [3] the following theorem:

Theorem 2.1. *Assume that the roots r_p, $p = 1, 2, \dots, m$ of the equation $r^m + \alpha_1 r^{m-1} + \dots + \alpha_m = 0$ satisfy the conditions $|r_p| < 1$, $p = 1, 2, \dots, m$. Then equation (1.1) in the space l_1 together with the initial conditions*

$$f(1) = u_1, f(2) = u_2, \dots, f(m) = u_m \tag{2.8}$$

is equivalent to the following operator equation in H_1:

$$f = \Gamma^{-1}u + \Gamma^{-1}V^m \left(\sum_{i=1}^{N} N_i(f) + \sum_{j=1}^{M} K_j(f) \right) , \tag{2.9}$$

where

$$\begin{aligned} u =& u_1 e_1 + (\alpha_1 u_1 + u_2) e_2 + \dots \\ &+ (\alpha_{m-1} u_1 + \alpha_{m-2} u_2 + \dots + u_m) e_m . \end{aligned} \tag{2.10}$$

Proof. Equation (2.6) can be written as follows:

$$V^{*m}(\Gamma f - \sum_{i=1}^{N} V^m N_i(f) - \sum_{j=1}^{M} V^m K_j(f)) = 0 \tag{2.11}$$

The null space of V^{*m} is spanned by the elements $e_1, e_2, \dots, e_m$. Thus we obtain from (2.11):

$$\Gamma f - \sum_{i=1}^{N} V^m N_i(f) - \sum_{j=1}^{M} V^m K_j(f)) = c_1 e_1 + c_2 e_2 + \dots + c_m e_m .$$

From (2.4) and (2.9) we find easily that:

$$c_1 = u_1, c_2 = \alpha_1 u_1 + u_2, \dots, c_m = u_m + \dots + u_2 \alpha_{m-2} + u_1 \alpha_{m-1}$$

and the theorem follows from the properties of the operator Γ. □

3. Existence and uniqueness theorem

In this Section we shall prove a theorem which predicts a unique solution of the nonlinear difference equation (1.1) in l_1, which satisfies the initial conditions (1.2). The proof of the theorem is based on two lemmas. In the first lemma we give the abstract forms of the right-hand side (nonlinear) part of the difference equation (1.1) defined on H_1, and on the second lemma we prove that the nonlinear operators which are the abstract forms presented in the first lemma are Fréchet differentiable in H_1. Moreover we find the Fréchet derivative for each one of them. Combining these two lemmas together with Theorem 2.1 and the fixed point theorem of Earle and Hamilton [2] we obtain the following:

Theorem 3.1. *Assume that the complex sequences $\beta_p(n), c_i(n)$ and $d_j(n)$, $p = 1,\dots,m$, $i = 1,\dots,N$, $j = 1,\dots,M$ satisfy:*

$$\lim_{n\to\infty} \beta_p(n) = 0\,, \quad \sup_n |c_i(n)| \le \gamma_i, \quad \sup_n |d_j(n)| \le \delta_j\,, \tag{3.1}$$

and the roots of the equation $r^m + \alpha_1 r^{m-1} + \dots + \alpha_m = 0$ satisfy the conditions $|r_p| < 1$, $p = 1, 2, \dots, m$. Then there exist positive numbers R_0 and P_0 such that for

$$|u| = |u_1| + |\alpha_1 u_1 + u_2| + \dots + |\alpha_{m-1} u_1 + \alpha_{m-2} u_2 + \dots + u_m| < P_0 \tag{3.2}$$

the equation

$$f(n+m) + \sum_{p=1}^{m} (\alpha_p + \beta_p(n)) f(n+m-p) \tag{3.3}$$
$$= \sum_{i=1}^{N} c_i(n) f(n+q_{i1}) f(n+q_{i2}) + \sum_{j=1}^{M} d_j(n) f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5})$$

together with the initial conditions

$$f(1) = u_1, f(2) = u_2, \dots, f(m) = u_m \tag{3.4}$$

where α_p, u_p are in general complex numbers, m, N, M are positive integers, q_{i1}, q_{i2}, $i = 1,\dots,N$, q_{j3}, q_{j4}, q_{j5}, $j = 1,\dots,M$ are non-negative integers, has a unique solution $f(n)$ in l_1. Moreover

$$\sum_{n=1}^{\infty} |f(n)| < R_0\,. \tag{3.5}$$

Lemma 1. (i) *Consider the diagonal operators $C_i : H \to H$ such that:*

$$C_i : C_i e_n = c_i(n) e_n\,, \quad n = 1, 2, \dots, \quad i = 1, .., N \tag{3.6}$$

and the nonlinear operators N_i', which are defined on H_1 as follows:

$$N_i'(f) = (f, e_{n+q_{i1}})(f, e_{n+q_{i2}}) e_n = f(n+q_{i1}) f(n+q_{i2}) e_n\,, \tag{3.7}$$

$1 \le i \le N$. Then the nonlinear operators:

$$N_i : N_i(f) = c_i(n) N_i'(f)\,, \quad i = 1, \dots, N \tag{3.8}$$

are defined in H_1 and are the abstract forms of the operators:

$$G_i : G_i(f(n+q_{i1}), f(n+q_{i2})) = c_i(n) f(n+q_{i1}) f(n+q_{i2})\,, \quad i = 1, \dots, N \tag{3.9}$$

in l_1.

(ii) *Consider the diagonal operators $D_j : H \to H$, such that:*

$$D_j : D_j e_n = d_j(n) e_n\,, \quad n = 1, 2, \ldots\,, \; j = 1, .., M \tag{3.10}$$

and the nonlinear operators K'_j, which are defined on H_1 as follows:

$$\begin{aligned} K'_j(f) &= (f, e_{n+q_{j3}})(f, e_{n+q_{j4}})(f, e_{n+q_{j5}}) e_n \\ &= f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5}) e_n\,, \end{aligned} \tag{3.11}$$

$1 \le j \le M$. Then the nonlinear operators:

$$K_j : K_j(f) = d_j(n) K_j(f)\,, \quad j = 1, \ldots, M \tag{3.12}$$

are defined in H_1 and are the abstract forms of the operators:

$$\begin{aligned} &F_j : F_j(f(n+q_{j3}), f(n+q_{j4}), f(n+q_{j5})) \\ &\quad = d_j(n) f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5})\,, \quad j = 1, \ldots, M \end{aligned} \tag{3.13}$$

in l_1.

Proof. (i) From (3.7) we have:

$$\begin{aligned} \|N'_i(f)\|_1 &= \sum_{r=1}^{\infty} |(N'_i(f), e_r)| = \sum_{r=1}^{\infty} |(f, e_{n+q_{i1}})(f, e_{n+q_{i2}})(e_n, e_r)| \\ &\Rightarrow \|N'_{ik}(f)\|_1 = |(f, e_{n+q_{i1}})| \cdot |(f, e_{n+q_{i2}})| \le \|f\|_1^2 \end{aligned} \tag{3.14}$$

Since $\|C_i\|_1 = \sup\limits_n |c_i(n)| \le \gamma_i$ we obtain from (3.8) and (3.14):

$$\|N_i(f)\|_1 \le \gamma_i \|f\|_1^2 < \infty\,, \tag{3.15}$$

since $f \in H_1$. Thus the nonlinear operators (3.8) are defined in H_1. Moreover we see that

$$\begin{aligned} &(N_i(f), e_n) = (c_i(n) f(n+q_{i1}) f(n+q_{i2}) e_n, e_n) \\ &\qquad \Rightarrow (N_i(f), e_n) = c_i(n) f(n+q_{i1}) f(n+q_{i2}) = G_i(f(n+q_{i1}), f(n+q_{i2}))\,, \end{aligned}$$

which means that N_i are the abstract forms of G_i in l_1.

(ii) Similarly from (3.11) we have:

$$\|K'_j(f)\|_1 \le \|f\|_1^3 \tag{3.16}$$

Since $\|D_j\|_1 = \sup\limits_n |d_j(n)| \le \delta_j$ we obtain from (3.12) and (3.16):

$$\|K_j(f)\|_1 \le \delta_{tk} \|f\|_1^3 < \infty\,. \tag{3.17}$$

Thus the nonlinear operators (3.12) are defined in H_1. Moreover we see that

$$\begin{aligned} &(K_j(f), e_n) = (d_j(n) f(n+q_{j3}) f(n+q_{j4}) f(n+q_{j5}) e_n, e_n) \\ &\qquad \Rightarrow (K_j(f), e_n) = F_j(f(n+q_{j3}), f(n+q_{j4}), f(n+q_{j5}))\,, \end{aligned}$$

which means that K_j are the abstract forms of F_j. $\square$

Lemma 2. (i) *The nonlinear operators (3.8) are Fréchet differentiable in* H_1 *and their Fréchet derivatives at the point* $f_0 \in H_1$ *are*

$$A_i(f_0)f = c_i(n)[(f, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) + (f_0, e_{n+q_{i1}})(f, e_{n+q_{i2}})]e_n\,. \tag{3.18}$$

(ii) *The nonlinear operators (3.12) are Fréchet differentiable in* H_1 *and their Fréchet derivatives at the point* $f_0 \in H_1$ *are*

$$\begin{aligned} A_j(f_0)f = d_j(n)[&(f, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f_0, e_{n+q_{j5}}) \\ &+ (f_0, e_{n+q_{j3}})(f, e_{n+q_{j4}})(f_0, e_{n+q_{j5}}) \\ &+ (f_0, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f, e_{n+q_{j5}})]e_n\,. \end{aligned} \tag{3.19}$$

Proof. (i) We shall first prove that the linear operators:

$$A_i'(f_0)f = [(f, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) + (f_0, e_{n+q_{i1}})(f, e_{n+q_{i2}})]e_n$$

are the Fréchet derivatives of the nonlinear operators (3.7) at the point $f_0 \in H_1$.

Indeed, A_i' are bounded operators for $f_0 \in H_1$, since:

$$\|A_i'(f_0)f\|_1 = \sum_{r=1}^{\infty} |(A_i'(f_0)f, e_r)|$$

$$= \sum_{r=1}^{\infty} |([(f, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) + (f_0, e_{n+q_{i1}})(f, e_{n+q_{i2}})]e_n, e_r)|$$

$$\Rightarrow \|A_i'(f_0)f\|_1 \le |(f, e_{n+q_{i1}})| \cdot |(f_0, e_{n+q_{i2}})| + |(f_0, e_{n+q_{i1}})| \cdot |(f, e_{n+q_{i2}})|$$

$$\Rightarrow \|A_i'(f_0)f\|_1 \le 2\|f_0\|_1\|f\|_1$$

and $\|f_0\|_1 < \infty$, since $f_0 \in H_1$. Also for $f_0 \in H_1$ and $h \in H_1$, we have:

$$\begin{aligned} \|N_i'(f_0+h) - N_i'(f_0) - A_i'(f_0)h\|_1 &= \sum_{r=1}^{\infty} |(N_i'(f_0+h) - N_i'(f_0) - A_i'(f_0)h, e_r)| \\ &= |(f_0+h, e_{n+q_{i1}})(f_0+h, e_{n+q_{i2}}) - (f_0, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) \\ &\quad -(h, e_{n+q_{i1}})(f_0, e_{n+q_{i2}}) - (f_0, e_{n+q_{i1}})(h, e_{n+q_{i2}})| \end{aligned}$$

$$\begin{aligned} = |[f_0(n+q_{i1}) + h(n+q_{i1})][f_0(n+q_{i2}) + h(n+q_{i2})] - f_0(n+q_{i1})f_0(n+q_{i2}) \\ -h(n+q_{i1})f_0(n+q_{i2}) - f_0(n+q_{i1})h(n+q_{i2})| \end{aligned}$$

$$\Rightarrow \|N_i'(f_0+h) - N_i'(f_0) - A_i'(f_0)h\|_1 = |h(n+q_{i1})| \cdot |h(n+q_{i2})|$$

$$\Rightarrow \|N_i'(f_0+h) - N_i'(f_0) - A_i'(f_0)h\|_1 = |(h, e_{n+q_{i1}})| \cdot |(h, e_{n+q_{i2}})| \le \|h\|_1^2$$

$$\Rightarrow \frac{\|N_i'(f_0+h) - N_i'(f_0) - A_i'(f_0)h\|_1}{\|h\|_1} \le \|h\|_1 \to 0$$

for $||h||_1 \to 0$. Thus the linear operators $A_i'(f_0)f$ are the Fréchet derivatives of the nonlinear operators (3.7) at the point $f_0 \in H_1$. Now we can prove that the linear operators (3.18) are the Fréchet derivatives of the nonlinear operators (3.8) at the point $f_0 \in H_1$. Indeed, the linear operators (3.18) are bounded, since:

$$||A_i(f_0)f||_1 \leq |c_i(n)| \cdot ||A_i'(f_0)f||_1 \leq 2\gamma_i ||f_0||_1 ||f||_1 .$$

Also

$$\begin{aligned} &||N_i(f_0+h) - N_i(f_0) - A_i(f_0)h||_1 \leq \gamma_i ||N_i'(f_0+h) - N_i'(f_0) - A_i'(f_0)h||_1 \\ &\Rightarrow \frac{||N_i(f_0+h) - N_i(f_0) - A_i(f_0)h||_1}{||h||_1} \leq \gamma_i ||h||_1 \to 0 \end{aligned}$$

for $||h||_1 \to 0$. Thus the linear operators (3.18) are the Fréchet derivatives of the nonlinear operators (3.8) at the point $f_0 \in H_1$.

(ii) Similarly, we shall first prove that the linear operators:

$$\begin{aligned} A_j'(f_0)f = [&(f, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f_0, e_{n+q_{j5}}) \\ &+(f_0, e_{n+q_{j3}})(f, e_{n+q_{j4}})(f_0, e_{n+q_{j5}}) + (f_0, e_{n+q_{j3}})(f_0, e_{n+q_{j4}})(f, e_{n+q_{j5}})]e_n \end{aligned}$$

are the Fréchet derivatives of the nonlinear operators (3.11) at the point $f_0 \in H_1$.

Indeed, A_j' are bounded operators for $f_0 \in H_1$, since:

$$||A_j'(f_0)f||_1 \leq 3||f_0||_1^2 ||f||_1$$

and $||f_0||_1 < \infty$, since $f_0 \in H_1$. Also for $f_0 \in H_1$ and $h \in H_1$, we have:

$$\frac{||K_j'(f_0+h) - K_j'(f_0) - A_j'(f_0)h||_1}{||h||_1} \leq 3||f_0||_1^2 ||h||_1 + ||h||_1^2 \to 0$$

for $||h||_1 \to 0$. Thus the linear operators $A_j'(f_0)f$ are the Fréchet derivatives of the nonlinear operators (3.11) at the point $f_0 \in H_1$. Now we can prove that the linear operators (3.19) are the Fréchet derivatives of the nonlinear operators (3.12) at the point $f_0 \in H_1$. Indeed, the linear operators (3.19) are bounded, since:

$$||A_j(f_0)f||_1 \leq |d_j(n)| \cdot ||A_j'(f_0)f||_1 \leq 3\delta_j ||f_0||_1 \cdot ||f||_1 .$$

Also

$$\begin{aligned} &||K_j(f_0+h) - K_j(f_0) - A_j(f_0)h||_1 \leq \delta_i ||K_j'(f_0+h) - K_j'(f_0) - A_j'(f_0)h||_1 \\ &\Rightarrow \frac{||K_j(f_0+h) - K_j(f_0) - A_j(f_0)h||_1}{||h||_1} \leq \delta_j (3||f_0||_1 \cdot ||h||_1 + ||h||_1^2 \to 0 \end{aligned}$$

for $||h||_1 \to 0$. Thus the linear operators (3.19) are the Fréchet derivatives of the nonlinear operators (3.12) at the point $f_0 \in H_1$. $\square$

Proof of Theorem 3.1. Consider the function:

$$\phi : \phi(f) = \Gamma^{-1}u + \sum_{i=1}^{N} \Gamma^{-1}V^m N_i(f) + \sum_{j=1}^{M} \Gamma^{-1}V^m K_j(f) \tag{3.20}$$

and assume that

$$\|\Gamma^{-1}\|_1 \leq L\,. \tag{3.21}$$

By Lemma 1 we have from (3.15) and (3.17):

$$\|\phi(f)\|_1 \leq L(|u| + \gamma_i \|f\|_1^2 + \delta_j \|f\|_1^3)\,. \tag{3.22}$$

Let $\|f\|_1 \leq \bar{R}$, where $\bar{R}$ can be as large as we want but not infinite. Then for $\|f\|_1 \leq R \leq \bar{R}$ we have from (3.22):

$$\|\phi(f)\|_1 \leq L(|u| + \gamma_i R^2 + \delta_j R^3)\,. \tag{3.23}$$

By hypothesis R is very large so there exists $R_1 \in [0, \bar{R}]$ such that

$$LR_1(\gamma_i + \delta_j R_1) > 1\,.$$

Thus the function

$$P(R) = 1 - LR(\gamma_i + \delta_j R) \tag{3.24}$$

has a first zero R_2 between 0 and R_1 since: $P(0) = 1 > 0$ and

$$P(R_1) = 1 - LR_1(\gamma_i + \delta_j R_1) < 0\,.$$

So the continuous function

$$P_1(R) = L^{-1}RP(R) \tag{3.25}$$

satisfies $P_1(0) = P_1(R_2) = 0$ and therefore attains a maximum at a point $R_0 : 0 < R_0 < R_2$. Now for every $\epsilon > 0, R = R_0$ and

$$|u| \leq P_1(R_0) - \frac{\epsilon}{L} \tag{3.26}$$

we find from (3.23):

$$\|\phi(f)\|_1 \leq R_0 - \epsilon < R_0 \tag{3.27}$$

for $\|f\|_1 < R_0$. This means that for

$$|u| < P_1(R_0) = P_0 \tag{3.28}$$

ϕ is a holomorphic map from $B(0, R_0)$ strictly inside $B(0, R_0)$. Indeed, it is obvious that $\phi(B(0, R_0)) \subset B(0, R_0)$ and $\phi(B(0, R_0))$ lies strictly inside $B(0, R_0)$, since if $k \in H_1 - B(0, R_0) \Rightarrow \|k\|_1 \geq R_0$ and $g \in \phi(B(0, R_0))$, i.e. there exists an $f \in B(0, R_0) \Rightarrow \|f\|_1 < R_0$ such that $\phi(f) = g$, then we find easily that $\|g - k\|_1 \geq \epsilon > \frac{\epsilon}{2} = \epsilon_1$. Thus applying the fixed point theorem of Earle and Hamilton [2] we find that equation $\phi(f) = f$ has a unique fixed point in H_1. This means equivalently that the initial value problem (3.3), (3.4) has a unique solution in l_1. □

Remark 3.1. Theorem 3.1 predicts local asymptotic stability. However, it is not a purely local result. The proof of the theorem has a constructive character and gives some information about the size of the region of attraction. This is something that one cannot obtain with the classical methods of linearisation [4]. Moreover, we can see in particular cases how the size of the above region depends on the parameters of the equation. Note that the region of attraction may be so small relative to a given application that practically the centre of this region could be considered as unstable. Also an equilibrium point of a difference equation could be unstable, but a very small neighborhood of it could be an attractor and thus from a practical point of view, it could be considered as stable [4].

Remark 3.2. The solution that is established by Theorem 3.1 is an element of l_1 and thus $\lim_{n\to\infty} f(n) = 0$. Also from (3.5) we obtain:

$$|f(n)| \leq \sum_{n=1}^{\infty} |f(n)| < R_0 \Rightarrow |f(n)| < R_0 .$$

This means that the solution that is predicted by Theorem 3.1 is a bounded solution of the initial value problem (3.3), (3.4) and R_0 is an upper bound of it.

Remark 3.3. Theorem 3.1 predicts a unique solution $\{f(n)\}$ of (3.3) in l_1. This means that $\lim_{n\to\infty} f(n) = 0$. Thus zero is a locally asymptotically stable equilibrium point of equation (3.3) with region of attraction given by (3.2). In the case where equation (3.3) has nonzero equilibrium points (ϱ), we set:

$$f(n) = g(n) + \varrho \tag{3.29}$$

and we apply Theorem 3.1 to the new equation which results from equation (3.3) after the transformation (3.29). If Theorem 3.1 can be applied to this transformed equation, then this equation has a unique solution in l_1, i.e. $g(n) \in l_1 \Rightarrow \lim_{n\to\infty} g(n) = 0$ and zero is a locally asymptotically stable equilibrium point with region of attraction given by (3.2). As a consequence, equation (3.3) has a unique solution, not in l_1, but in the space $\{\varrho\} + l_1$. Also $\lim_{n\to\infty} f(n) = \varrho$ and thus ϱ is a locally asymptotically stable equilibrium point. Its region of attraction results from the region of attraction for the zero equilibrium point of the

transformed equation using (3.29). Finally the upper bound of f(n) is given by
$$|f(n)| < R_0 + |\varrho|$$
Remark 3.4. Theorem 3.1 holds also in the case where $n = 0, 1, 2, \ldots$. In this case the orthonormal basis of H consists of the elements $e_0, e_1, e_2, \ldots$ and the shift operator and its adjoint are defined as follows:

$$V : Ve_n = e_{n+1}\,,\ \ n = 0, 1, 2, \ldots$$

$$V^* : V^* e_n = e_{n-1}\,,\ \ n \geq 1\,, V^* e_0 = 0\,.$$

Also the solution $f(n) \in l_1$ of the nonlinear difference equation (3.3) does not satisfy conditions (3.4) anymore, but instead satisfy the following conditions:

$$f(0) = u_0, f(1) = u_1, f(2) = u_2, \ldots, f(m-1) = u_{m-1}\,.$$

Finally the relationship (3.2) should be replaced by the following:

$$|u| = |u_0| + |\alpha_1 u_0 + u_1| + \cdots + |\alpha_{m-1} u_0 + \alpha_{m-2} u_1 + \cdots + u_{m-1}| < P_0\,,$$

where $u = u_0 e_0 + (\alpha_1 u_0 + u_1) e_1 + \cdots + (\alpha_{m-1} u_0 + \alpha_{m-2} u_1 + \cdots + u_{m-1}) e_{m-1}$.

4. Examples

1) Consider the difference equation:

$$f(n+1) + \alpha(n) f(n) = d(n) f(n+2) f(n+1) f(n)\,, \tag{4.1}$$

where $\lim\limits_{n\to\infty} \alpha(n) = \alpha$, $|\alpha| < 1$, $\sup\limits_n |d(n)| \leq \delta$ and $\alpha(n), d(n)$ are in general complex-valued sequences, α is in general a complex number and δ is a real positive number.

First, we shall show that zero is a locally asymptotically stable equilibrium point of equation (4.1). Moreover we shall find a region of attraction for it, a bound for the solution of equation (4.1) and the radius of convergence for the generating function $f(z)$.

Equation (4.1) can be written as follows:

$$f(n+1) + \alpha f(n) + [\alpha(n) - \alpha] f(n) = d(n) f(n+2) f(n+1) f(n)\,. \tag{4.2}$$

This equation results from equation (3.3) for:

$$m = 1, \alpha_1 = \alpha, \beta_1(n) = \alpha(n) - \alpha, c_i(n) \equiv 0, \gamma_i \equiv 0\,,$$
$$M = 1, d_1(n) = d(n), \delta_1 = \delta, q_{13} = 2, q_{14} = 1, q_{15} = 0\,.$$

In this case the operator Γ has the form

$$\Gamma = I + \alpha V + V D_1\,,\ \text{where } D_1 e_n = [\alpha(n) - \alpha] e_n$$

and let $\|\Gamma^{-1}\|_1 \leq L$. Also $P(R) = 1 - L\delta R^2$ and $P_1(R) = \frac{R}{L} - \delta R^3$. It follows easily that the region of attraction predicted by Theorem 3.1 is given by:

$$|f(0)| < \frac{2}{3L\sqrt{3\delta L}}\,. \tag{4.3}$$

Thus zero is a locally asymptotically stable equilibrium point of equation (4.1) with region of attraction given by (4.3). Also $f(n)$ is bounded and in particular the following holds:

$$|f(n)| < \frac{1}{\sqrt{3\delta L}}\,. \tag{4.4}$$

Finally from equation (4.1) we have:

$$\frac{f(n+1)}{f(n)} + \alpha(n) = d(n)f(n+2)f(n+1)\,.$$

The solution $\{f(n)\}$ predicted by Theorem 3.1 is an element of l_1 and so $\lim_{n\to\infty} f(n) = 0$. Thus

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = -\alpha\,.$$

This means that for every solution $\{f(n)\}$ starting at a point given by (4.3), the generating analytic function $f(z) = \sum_{n=1}^{\infty} f(n)z^{n-1}$ converges absolutely for $|z| < \frac{1}{|\alpha|}$.

If $\alpha(n) = \alpha$, $|\alpha| < 1$, α complex and $d(n) = d$, d complex, then the equilibrium point zero is a locally asymptotically stable equilibrium point of equation (4.1) with region of attraction given by:

$$|f(0)| < \frac{2}{3|\alpha|\sqrt{3|\alpha|\cdot|d|}}\,, \tag{4.5}$$

an upper bound of $f(n)$ is $\frac{1}{\sqrt{3|\alpha|\cdot|d|}}$, i.e.

$$|f(n)| < \frac{1}{\sqrt{3|\alpha|\cdot|d|}} \tag{4.6}$$

and the corresponding generating analytic function $f(z) = \sum_{n=1}^{\infty} f(n)z^{n-1}$ converges absolutely for $|z| < \frac{1}{|\alpha|}$.

Let us consider now the case where

$$\alpha(n) = \alpha\,,\ d(n) = d\,,$$

α, d real numbers. Then equation (4.1) can be written as follows:

$$f(n+1) + \alpha f(n) = df(n+2)f(n+1)f(n)\,. \tag{4.7}$$

The nonzero real equilibrium points (ϱ) of equation (4.7) are:

$$\varrho_1 = \sqrt{\frac{1+\alpha}{d}},\ \varrho_2 = -\sqrt{\frac{1+\alpha}{d}},\ (1+\alpha)d > 0\,. \tag{4.8}$$

In the following we shall show that ϱ_1, ϱ_2 are locally asymptotically stable equilibrium points of (4.7). Moreover we shall find a region of attraction for each one of them, a bound of the solution of equation (4.1) and the radius of convergence for the corresponding generating analytic function. We set $f(n) = g(n) + \varrho$ and equation (4.7) becomes:

$$\begin{aligned} g(n+2) &+ \frac{\alpha}{1+\alpha} g(n+1) + \frac{1}{1+\alpha} g(n) \\ &= -\frac{d}{1+\alpha} g(n+2)g(n+1)g(n) - \frac{d\varrho}{1+\alpha} g(n+2)g(n+1) \\ &\quad - \frac{d\varrho}{1+\alpha} g(n+2)g(n) - \frac{d\varrho}{1+\alpha} g(n+1)g(n)\,. \end{aligned} \tag{4.9}$$

We shall consider the following three cases for the values of α and d:
i) $0 < \alpha < 2 + 2\sqrt{2}, d > 0$,
ii) $\alpha \geq 2 + 2\sqrt{2}, d > 0$,
iii) $\alpha < -1, d < 0$.

In the first case the operator Γ has the form

$$\Gamma = (I - r_1 V)(I - r_2 V)\,,$$

where $r_{1,2} = \frac{-\alpha \pm i\sqrt{4+4\alpha-\alpha^2}}{2(1+\alpha)}$ and $|r_{1,2}| = \frac{1}{\sqrt{1+\alpha}} < 1$. In this case $L = (1 - \frac{1}{\sqrt{1+\alpha}})^{-2}$. As before we find:

$$\begin{aligned} P(R) &= 1 - LR(3\sqrt{\frac{d}{1+\alpha}} + \frac{Rd}{1+\alpha})\,, \\ P_1(R) &= L^{-1}R(1 - LR(3\sqrt{\frac{d}{1+\alpha}} + \frac{Rd}{1+\alpha})\,. \end{aligned}$$

Thus point zero in each case ($\varrho = \varrho_1$, $\varrho = \varrho_2$) is locally asymptotically stable equilibrium points of equation (4.9) with region of attraction given by:

$$\begin{aligned} |g(0)| + \left| \frac{\alpha}{1+\alpha} g(0) - g(1) \right| &< \frac{10 + 8\alpha - 4\sqrt{1+\alpha}}{3(1+\alpha)} \sqrt{\frac{4(1+\alpha)}{3d} + \frac{1}{3d} - \frac{2\sqrt{1+\alpha}}{3d}} \\ &\quad -3\sqrt{\frac{1+\alpha}{d}} - \frac{1}{\sqrt{d}\sqrt{1+\alpha}} + 2\,. \end{aligned} \tag{4.10}$$

Also $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| < \sqrt{\frac{4(1+\alpha)}{3d} + \frac{1}{3d} - \frac{2\sqrt{1+\alpha}}{3d}} - \sqrt{\frac{1+\alpha}{d}}\,. \tag{4.11}$$

Thus the equilibrium points ϱ_1, ϱ_2 are locally asymptotically stable with regions of attraction given respectively by:

$$\begin{aligned}&\left|f(0)-\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)-\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &\quad<-3\sqrt{\frac{1+\alpha}{d}}-\frac{1}{\sqrt{d}\sqrt{1+\alpha}}+2\\ &\qquad+\frac{10+8\alpha-4\sqrt{1+\alpha}}{3(1+\alpha)}\sqrt{\frac{4(1+\alpha)}{3d}+\frac{1}{3d}-\frac{2\sqrt{1+\alpha}}{3d}}\,.\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}&\left|f(0)+\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)+\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &\quad<-3\sqrt{\frac{1+\alpha}{d}}-\frac{1}{\sqrt{d}\sqrt{1+\alpha}}+2\\ &\qquad+\frac{10+8\alpha-4\sqrt{1+\alpha}}{3(1+\alpha)}\sqrt{\frac{4(1+\alpha)}{3d}+\frac{1}{3d}-\frac{2\sqrt{1+\alpha}}{3d}}\,.\end{aligned}\tag{4.13}$$

Also $\{f(n)\}$ is bounded and in particular the following holds:

$$|f(n)|<\sqrt{\frac{4(1+\alpha)}{3d}+\frac{1}{3d}-\frac{2\sqrt{1+\alpha}}{3d}}\,.\tag{4.14}$$

In the second case the operator Γ has the form

$$\Gamma=(I-r_1V)(I-r_2V)\,,$$

where $r_1=\frac{-\alpha+\sqrt{\alpha^2-4\alpha-4}}{2(1+\alpha)}<0$, $r_2=\frac{-\alpha-\sqrt{\alpha^2-4\alpha-4}}{2(1+\alpha)}<0$ and $|r_{1,2}|<1$. In this case $L=\frac{1+\alpha}{2}$. As before we find:

$$P(R)=1-\frac{3R\sqrt{d(1+\alpha)}}{2}-\frac{dR^2}{2}\,,$$

$$P_1(R)=\frac{2R}{1+\alpha}-\frac{3\sqrt{d}R^2}{\sqrt{1+\alpha}}-\frac{dR^3}{1+\alpha}\,.$$

Thus point zero in each case ($\varrho=\varrho_1$, $\varrho=\varrho_2$) is locally asymptotically stable equilibrium point of equation (4.9) with region of attraction given by:

$$|g(0)|+\left|\frac{\alpha}{1+\alpha}g(0)-g(1)\right|<\frac{2}{\sqrt{d}}\left[\frac{(\alpha+\frac{5}{3})^{\frac{3}{2}}}{1+\alpha}-\frac{5\alpha+8}{\sqrt{1+\alpha}}\right]\,.\tag{4.15}$$

Also $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| < \frac{\sqrt{\alpha + \frac{5}{3}} - \sqrt{\alpha + 1}}{\sqrt{d}} \,. \tag{4.16}$$

Thus the equilibrium points ϱ_1, ϱ_2 are locally asymptotically stable with regions of attraction given respectively by:

$$\begin{aligned} &\left| f(0) - \sqrt{\frac{1+\alpha}{d}} \right| + \left| \frac{\alpha}{1+\alpha} f(0) + f(1) - \frac{2\alpha + 1}{\sqrt{d(1+\alpha)}} \right| \\ &< \frac{2}{\sqrt{d}} \left[\frac{(\alpha + \frac{5}{3})^{\frac{3}{2}}}{1+\alpha} - \frac{5\alpha + 8}{\sqrt{1+\alpha}} \right] . \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} &\left| f(0) + \sqrt{\frac{1+\alpha}{d}} \right| + \left| \frac{\alpha}{1+\alpha} f(0) + f(1) + \frac{2\alpha + 1}{\sqrt{d(1+\alpha)}} \right| \\ &< \frac{2}{\sqrt{d}} \left[\frac{(\alpha + \frac{5}{3})^{\frac{3}{2}}}{1+\alpha} - \frac{5\alpha + 8}{\sqrt{1+\alpha}} \right] . \end{aligned} \tag{4.18}$$

Also $\{f(n)\}$ is bounded and in particular the following holds:

$$|f(n)| < \frac{\sqrt{\alpha + \frac{5}{3}}}{\sqrt{d}} \,. \tag{4.19}$$

In the third case the operator Γ has the form

$$\Gamma = (I - r_1 V)(I - r_2 V) \,,$$

where $r_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\alpha - 4}}{2(1+\alpha)} > 0$, $r_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\alpha - 4}}{2(1+\alpha)} > 0$ and $|r_{1,2}| < 1$. In this case $L = \frac{1}{2}$. As before we find:

$$P(R) = 1 - \frac{3R}{2}\sqrt{\frac{d}{1+\alpha}} - \frac{dR^2}{2(1+\alpha)} \,,$$

$$P_1(R) = 2R - 3\sqrt{\frac{d}{1+\alpha}} R^2 - \frac{dR^3}{1+\alpha} \,.$$

Thus point zero in each case ($\varrho = \varrho_1$, $\varrho = \varrho_2$) is locally asymptotically stable equilibrium point of equation (4.9) with region of attraction given by:

$$|g(0)| + \left| \frac{\alpha}{1+\alpha} g(0) - g(1) \right| < \frac{10\sqrt{15} - 36}{9} \sqrt{\frac{1+\alpha}{d}} \,. \tag{4.20}$$

Also $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| < \frac{\sqrt{15}-3}{3}\sqrt{\frac{1+\alpha}{d}}\,. \tag{4.21}$$

Thus the equilibrium points ϱ_1, ϱ_2 are locally asymptotically stable with regions of attraction given respectively by:

$$\begin{aligned}&\left|f(0)-\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)-\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &\quad<\frac{10\sqrt{15}-36}{9}\sqrt{\frac{1+\alpha}{d}}\,.\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\left|f(0)+\sqrt{\frac{1+\alpha}{d}}\right|+\left|\frac{\alpha}{1+\alpha}f(0)+f(1)+\frac{2\alpha+1}{\sqrt{d(1+\alpha)}}\right|\\ &\quad<\frac{10\sqrt{15}-36}{9}\sqrt{\frac{1+\alpha}{d}}\,.\end{aligned} \tag{4.23}$$

Also $\{f(n)\}$ is bounded and in particular the following holds:

$$|f(n)| < \frac{\sqrt{15}}{3}\sqrt{\frac{1+\alpha}{d}}\,. \tag{4.24}$$

Finally from equation (4.8) we have (for all these cases):

$$\frac{f(n+1)}{f(n)}+\alpha = df(n+2)f(n+1)\,. \tag{4.25}$$

The solution $\{g(n)\}$ of equation (4.9) which is predicted by Theorem 3.1 is an element of l_1 and so $\lim_{n\to\infty} g(n) = 0$. Thus $\lim_{n\to\infty} f(n) = \varrho$ and $f(n)$ is not an element of l_1 but it is an element of the space of bounded complex sequences. So from (4.25) we have:

$$\lim_{n\to\infty}\frac{f(n+1)}{f(n)} = -\alpha + d\frac{1+\alpha}{d} = 1\,.$$

This means that for every solution $\{f(n)\}$ starting at a point given by (4.12), (4.13) or (4.17), (4.18) or (4.22), (4.23), depending on the values of α and d, the corresponding generating analytic function $f(z)=\sum_{n=1}^{\infty} f(n)z^{n-1}$ converges absolutely for $|z|<1$.

Remark 4.1. The method used in this paper does not give us any information in the case where $-1<\alpha\le 0$, as in this case the operator Γ has the form $\Gamma=(I-r_1V)(I-r_2V)$ and both r_1, r_2 are in modulus greater than 1.

Remark 4.2. From (4.3), (4.5), (4.12), (4.13), (4.17), (4.18), (4.22) and (4.23) we see how each region of attraction depends on the parameters α and d of the equation.

Remark 4.3. From equation (4.1) for $\alpha(n) = A$ and $d(n) = 1$ we obtain the following difference equation:

$$f(n+2)f(n+1)f(n) = Af(n) + f(n+1)\,.$$

In the case when this difference equation has positive solutions with positive initial conditions it is known [5] that its positive equilibrium point is globally asymptotically stable for $A \in (0, \infty)$.

2) Consider the difference equation:

$$g(n+3)g(n+2)g(n+1) + g(n+3)g(n) = g(n+2) + g(n+1)g(n), \qquad n = 0, 1, \dots \tag{4.26}$$

The three equilibrium points (ϱ) of equation (4.26) are:

$$\varrho_1 = 0, \varrho_2 = 1, \varrho_3 = -1\,.$$

We shall show that the three equilibrium points of equation (4.26) are locally asymptotically stable. Moreover we shall find a region of attraction for each one of them and a bound of the solution of equation (4.26).

We set $g(n) = f(n) + \varrho$ into equation (4.26) and we obtain, after some manipulation:

$$\begin{aligned}(\varrho^2+\varrho)f(n+3) + (\varrho^2-1)f(n+2) + (\varrho^2-\varrho)f(n+1)& \\ = -\,f(n+3)f(n+2)f(n+1) - \varrho f(n+3)f(n+2)& \\ -\,\varrho f(n+3)f(n+1) - \varrho f(n+2)f(n+1)& \\ -\,f(n+3)f(n) + f(n+1)f(n)\,.&\end{aligned} \tag{4.27}$$

For the equilibrium point $\varrho_1 = 0$ equation (4.27) becomes:

$$f(n+2) = f(n+3)f(n+2)f(n+1) + f(n+3)f(n) - f(n+1)f(n)\,. \tag{4.28}$$

This equation results from equation (3.3) for:

$$m = 2, \alpha_1 = \alpha_2 = 0, \beta_1(n) = \beta_2(n) = 0\,,$$

$$N = 2, c_1(n) = 1, c_2(n) = -1, q_{11} = 3, q_{12} = q_{22} = 0, q_{21} = 1, \gamma_1 = \gamma_2 = 1$$

$$M = 1, d_1(n) = 1, q_{13} = 3, q_{14} = 2, q_{15} = 1, \delta_1 = 1\,.$$

In this case the operator Γ has the form:

$$\Gamma = I^2 = I \text{ and } \|\Gamma^{-1}\|_1 = 1 = L.$$

Also $P(R) = 1 - 2R - R^2$ and $P_1(R) = R - 2R^2 - R^3$.

It follows easily that the attraction region of the equilibrium point zero of equation (4.27), predicted by Theorem 3.1 is given by:

$$|f(0)| + |f(1)| < 0.11261179 \tag{4.29}$$

Thus $\varrho_1 = 0$ is a locally asymtotically stable equilibrium point of equation (4.26) and its region of attraction is given by:

$$|g(0)| + |g(1)| < 0.11261179 \tag{4.30}$$

Also $\{f(n)\}$ and thus $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| = |f(n)| < 0.215250437 \tag{4.31}$$

For the equilibrium point $\varrho_2 = 1$ equation (4.27) becomes:

$$\begin{aligned} f(n+3) = &-\frac{1}{2}f(n+3)f(n+2)f(n+1) - \frac{1}{2}f(n+3)f(n+2) \\ &-\frac{1}{2}f(n+3)f(n+1) - \frac{1}{2}f(n+2)f(n+1) \\ &-\frac{1}{2}f(n+3)f(n) + \frac{1}{2}f(n+1)f(n)\,. \end{aligned} \tag{4.32}$$

In a similar way it follows easily that the attraction region of the equilibrium point zero of equation (4.32), predicted by Theorem 3.1 is given by:

$$|f(0)| + |f(1)| + |f(2)| < 0.096322046 \tag{4.33}$$

Thus $\varrho_2 = 1$ is a locally asymtotically stable equilibrium point of equation (4.26) and its region of attraction is given by:

$$|g(0) - 1| + |g(1) - 1| + |g(2) - 1| < 0.096322046 \tag{4.34}$$

Also $\{f(n)\}$ and thus $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| = |f(n) + 1| \leq |f(n)| + 1 < 1.189254787 \tag{4.35}$$

For the equilibrium point $\varrho_3 = -1$ equation (4.27) becomes:

$$\begin{aligned} f(n+1) = &-\frac{1}{2}f(n+3)f(n+2)f(n+1) + \frac{1}{2}f(n+3)f(n+2) \\ &+\frac{1}{2}f(n+3)f(n+1) + \frac{1}{2}f(n+2)f(n+1) \\ &-\frac{1}{2}f(n+3)f(n) + \frac{1}{2}f(n+1)f(n)\,. \end{aligned} \tag{4.36}$$

In a similar way it follows easily that the attraction region of the equilibrium point zero of equation (4.36), predicted by Theorem 3.1 is given by:

$$|f(0)| < 0.096322046 \tag{4.37}$$

Thus $\varrho_3 = -1$ is a locally asymtotically stable equilibrium point of equation (4.26) and its region of attraction is given by:

$$|g(0)+1| < 0.096322046 \tag{4.38}$$

Also $\{f(n)\}$ and thus $\{g(n)\}$ is bounded and in particular the following holds:

$$|g(n)| = |f(n)-1| \leq |f(n)|+1 < 1.189254787 \tag{4.39}$$

Remark 4.4. In the case when equation (4.26) has positive solutions with positive initial conditions, it is proved recently in [1] that its positive equilibrium point is globally asymptotically stable.

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