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WIENER INTEGRAL IN THE SPACE OF SEQUENCES OF REAL NUMBERS

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Abstract. Let $i : H \to W$ be the canonical Wiener space where $W=\{\sigma : [0,T] \to \mathbb{R}$ continuous with $\sigma(0) = 0\}$, H is the Cameron-Martin space and i is the inclusion. We lift a isometry $H \to l_2$ to a linear isomorphism $\Phi : W \to \mathcal{V} \subset \mathbb{R}^\infty$ which pushes forward the Wiener structure into the abstract Wiener space (AWS) $i : l_2 \to \mathcal{V}$. Properties of the Wiener integration in this AWS are studied.

1. Introduction

In this article we are interested in a representation of the Wiener space (trajectories of the Brownian motion) in the space of sequences of real numbers. The classical Wiener space is the canonical example of the probability structure called abstract Wiener space. Before we state well known properties in this structure we recall that given a Banach space B endowed with a centered Gaussian probability measure μ on the Borel σ-algebra and a Hilbert space $H \subset B$ such that the inclusion $i : H \to B$ is a continuous linear injection with dense range, we call the triple (i, H, B) an *abstract Wiener space* (abreviatted AWS) if

$$\int_B e^{\sqrt{-1}l(x)}\mu\,(dx) = e^{-\frac{|l|_H^2}{2}}$$

for all $l \in B^*$, the dual space of B, where $|\cdot|_H$ denotes the norm in H. It will turn out that $B^* \subset H$ (see the comments after the next theorem) so that the formula above makes sense. In this context we say that the Hilbert space H *radonifies* the centered Gaussian measure μ on B. Note that if F is a finite dimensional space and $S : B \to F$ is a surjective linear map then the induced measure $S_*\mu$ on F is a centered Gaussian with respect to the inner product pushed forward by the composition $S \circ i : H \to F$.

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The canonical Wiener space $(W, \mathcal{F}, \mathbb{W})$ in the interval $[0, T]$ is the Banach space of continuous functions:

$$W = \{\omega : [0, T] \to \mathbb{R} \text{ continuous with } \omega(0) = 0\}$$

with the uniform norm endowed with the σ-algebra $\mathcal{F}$ generated by finite dimensional cylinders and the Gaussian probability measure $\mathbb{W}$ is such that the canonical process associated to this probability space is a homogeneous Markov process with transition probabilities given by the heat kernel

$$P_t(x, dy) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|y-x|^2}{2t}} dy\,.$$

The σ-algebra $\mathcal{F}$ coincides with the Borel σ-algebra. The following theorem guarantees that a centered Gaussian measures on a Banach space generates an AWS structure, see *e.g.* Gross (1970), Kuo (1975), Watanabe (1984), Bogachev (1996) and the references therein.

Theorem 1. [Segall, Kallianpur, Gross]. *Let $(B, \mathcal{G}, \mu)$ be a separable Banach space with a centered Gaussian probability measure μ on the Borel σ-algebra $\mathcal{G}$. Then, there exists a unique (up to isometry) separable Hilbert space and a continuous linear injection $i : (H, <\cdot, \cdot>_H) \to B$ which radonifies μ.*

Note that B^*, the dual space of B is contained in $L_2(B, \mu)$, so that the Hilbert space H is constructed as the L_2-closure of B^*. Moreover if $l \in B^* (\subset H^* \simeq H)$ then $l_*(\mu)$ is a Gaussian $N(0, |l|_H)$ on the real line. Among other interesting properties in this structure, we recall the Cameron-Martin formula: Let $T_b : B \to B$ be the translation $T_b(x) = x + b$ for $x \in B$, then the induced measure $(T_b)_* \mu$ on B is quasi-invariant with respect to the original μ if and only if $b \in i(H)$, moreover, writing $b = i(h)$, the Radon-Nikodyn derivative is given by:

$$\frac{d(T_b)_* \mu}{d\mu}(\omega) = e^{-\frac{|h|_H^2}{2} - \langle h, \omega \rangle^{\sim}}$$

where $\langle h, \cdot \rangle^{\sim} : B \to \mathbb{R}$ is a Gaussian random variable given by the Wiener integral (see section 3). A proof of this formula can be done as an application of the Girsanov theorem, see *e.g.* Revuz and Yor (1991). It is well known that the Hilbert space which radonifies the canonical Wiener space is the *Cameron-Martin* space defined by

$$H = \left\{ h(t) = \int_0^t h'(s)\, ds, \text{ with } h' \in L^2([0, T]) \right\}$$

with the inner product inherited by $L^2([0, T])$, *i.e.* $\langle h, g \rangle_H = \langle h', g' \rangle_{L^2}$, (see *e.g.* Watanabe (1984)).

In this article we intend to create an isomorphism from the canonical Wiener space W to an AWS constructed in the sequences of real number, that is, to push the whole AWS structure in some subspace of $\mathbb{R}^\infty$. This isomorphism establishes a new framework which, depending on the problem, provides a new mathematical interpretation. For instance the Wiener integration assumes a new interesting face when it is taken in this space (see section 3).

There are several different ways of representing W as a sequence of real numbers but in general the associated Hilbert space which radonifies the induced measure is not trivial. Take for instance $\{q_i\}_{i\geq 1} \subset [0,T]$ a dense countable subset, then, the map $\sigma : W \to l_\infty$ given by $(\omega(q_i))_{i\geq 1}$ is an injection. Obviously, if we push the Wiener measure to l_∞, it would be a Gaussian measure over an infinite dimensional closed subspace and by Theorem 1 there would exist a Hilbert space $\overline{H} \subset l_\infty$ which would provide an AWS $i : \overline{H} \to l_\infty$, however the Hilbert space $\overline{H}$ is not trivial.

The motivation to construct an isomorphism of AWS into a subspace $\mathcal{V} \subset \mathbb{R}^\infty$ starts mainly in the fact that via Fourier coefficients there is a canonical isometry between $L^2([0,T])$ and l_2 which also creates a natural isometry $\varphi : H \to l_2$ which sends basis to basis. So, we look for an isomorphism of W which extends this isometry to $\Phi : W \to \mathcal{V}$. Hence, providing $\mathcal{V}$ with the induced norm from W we will have the reproduction of the Wiener structure $i : l_2 \to \mathcal{V}$, with i being the inclusion of $l_2 \subset \mathcal{V}$. Note that, like in the canonical Wiener structure, here the Hilbert space l_2 (as a vector space) has two topologies: the inherited norm $(l_2, ||\cdot||_{\mathcal{V}})$ with respect to which l_2 is dense in $\mathcal{V}$, and $(l_2, <\cdot,\cdot>_{l_2})$ with respect to which l_2 is closed.

In the next section we construct the isomorphism using Fourier series technic. In the section 3 we consider the probability aspects of the Wiener isomorphism and establish the Wiener integration in the space of sequences.

2. The Isomorphism

Our method is based on the Fourier series of each trajectory of the Wiener process. In fact the isomorphism $\Phi : W \to \mathbb{R}^\infty$ is simply the coefficients of a formal derivative of the Fourier series; this procedure will guarantee that the elements of the Cameron-Martin space are mapped isomorphically into l_2. We shall divide the details and properties of these maps in two parts: firstly the Fourier series itself and secondly the formal derivative.

2.1. **The Fourier Series.** We shall write each continuous function $\omega \in W$ in a Fourier series of sines. We eliminate the cosine component using the analytical trick of extending ω with domain $[0,T]$ to a continuous $\overline{\omega}$ with domain $[0,4T]$. So, for a fixed trajectory $\omega : [0,T] \to \mathbb{R}$ let $\overline{\omega} : [0,4T] \to \mathbb{R}$ be the extension defined as follows:

$$\overline{\omega} = \begin{cases} \omega(t) & \text{if} \quad t \in [0,T] \\ \omega(2T-t) & \text{if} \quad t \in [T,2T] \\ -\omega(t-2T) & \text{if} \quad t \in [2T,3T] \\ -\omega(4T-t) & \text{if} \quad t \in [3T,4T]. \end{cases}$$

If we extend $\overline{\omega}$ periodically into the whole real line we would get an odd continuous function hence the cosine coefficients of the Fourier series vanish. There are several choices for an odd extension of ω, the extension $\overline{\omega}$ defined above has the advantage of symmetry (see the example at the end of section 3). Now, writing

$$a_n = a_n(\omega) = \sqrt{\tfrac{1}{2T}} \int_0^{4T} \overline{\omega}(s) \sin\left(\frac{\pi n}{2T} s\right) \, ds,$$

the Fourier analysis guarantees that the series

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n t}{2T}\right)$$

converges to $\overline{\omega}(t)$ in $L^p([0,4T])$ for $1<p<\infty$, hence, restricting the domain, it also converges to $\omega(t)$ in $L^p([0,T])$. The convergence is uniform if ω belongs to the subspace of bounded variation trajectories (of probability zero), in particular if ω is an element of the Cameron-Martin space H, see *e.g.* Katznelson (1976).

This procedure defines the linear isomorphism

$$\phi : W \longrightarrow U \subset l_2$$

where, given $\omega \in W$, $\phi(\omega) = (a_n(\omega))_{n\geq 1} \in l_2$. The linear operator ϕ is continuous, in fact it is the composition of the continuous injection $\left(W, ||\cdot||_{\sup}\right) \to L^2([0,T])$ and the Fourier isometry $L^2([0,T]) \to l_2$ (Bessel-Parseval formula). Moreover, since W is dense in $L^2([0,T])$ it implies that $\phi(W) = U$ is dense in l_2.

We remark that if $\omega \in H$ then its Fourier series converges absolutely, that is $a_n(\omega) \in l_1$, precisely, there is the following inequality which relates the three norms associated to ω:

$$||a_n(\omega)||_{l_1} \leq ||\omega||_{L^1[0,T]} + \left(2\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^2 ||\omega||_H \ .$$

See *e.g.* Katznelson (1976).

2.2. **The Formal Derivative.** If ω is an element in H, then its derivative ω' is in $L^2([0,T])$ and the derivative of the extension $(\overline{\omega})'$ is in $L^2([0,4T])$. Extending periodically the derivative $(\overline{\omega})'$ to the whole real line we get an even function, hence its Fourier series is a cosine series

$$\sum_{n=1}^{\infty} b_n \cos\left(\frac{\pi n t}{2T}\right)$$

where

$$b_n = b_n(\omega) = \sqrt{\tfrac{1}{2T}} \int_0^{4T} (\overline{\omega})'(s) \cos\left(\frac{\pi n}{2T} s\right) \, ds \, .$$

The integration by parts formula provides

$$\begin{aligned} b_n &= \sqrt{\tfrac{1}{2T}} \cdot \overline{\omega}(t) \cos\left(\frac{\pi n}{2T} t\right)\Big|_{t=0}^{t=4T} + \tfrac{\pi n}{2T} \cdot \sqrt{\tfrac{1}{2T}} \int_0^{4T} \overline{\omega}(s) \sin\left(\tfrac{\pi n}{2T} s\right) \, ds \\ &= \frac{\pi n}{2T} \cdot a_n \, , \end{aligned}$$

which shows that, independently of the uniform convergence of the series of term by term derivatives, if $\omega \in H$ then the Fourier series of ω' is given by term by term derivation of the Fourier series of ω.

Again, thinking of Fourier coefficients as a sequence of real number, the calculation above suggests us to define a formal derivative operator given by:

$$\begin{aligned} D : U \subset l_2 &\longrightarrow \mathcal{V} \subset \mathbb{R}^\infty \\ (a_n)_{n\geq 1} &\longmapsto \tfrac{\pi}{2T}(n \cdot a_n)_{n\geq 1}. \end{aligned}$$

Remark 1. *This operator can also be seen as the extension of (minus) the infinitesimal generator of the semigroup of contractions $T_t : l_2 \to l_2$ given by the diagonal operator $T_t(a_n) = (a_n e^{-nt})$. Its infinitesimal generator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to l_2$ is given by $\mathcal{A}(a_n) = (-na_n)$ when $(a_n) \in \mathrm{Dom}(\mathcal{A})$.*

Finally we define the isomorphism $\Phi : W \longrightarrow \mathcal{V} \subset \mathbb{R}^\infty$ by the composition $\Phi = D \circ \phi$. We push the norm and the Gaussian measure by this linear map from W to the subspace $\mathcal{V}$ to get the Gaussian structure $(\mathcal{V}, ||\cdot||_{\mathcal{V}}, \Phi_*(\mathbb{W}))$. By the isometries $H \approx L^2[(0,T)] \approx l_2$ we have that $D(\phi(H)) = l_2$. One easily sees that the inclusion $i : l_2 \to (\mathcal{V}, ||\cdot||_{\mathcal{V}}, \Phi_*(\mathbb{W}))$ is an AWS, hence Φ provides an isomorphism (of AWS) between the classical Wiener space $i : H \to W$ and the new strucuture constructed in space of sequences, which was the purpose of this paper.

Note that, by the same argument we have used to conclude that $\phi(H) \subset l_1$, if $\omega \in C^2$ then $\Phi(\omega) \in l_1$. We have that $l_p \subset \mathcal{V}$ for all $1 \leq p \leq \infty$; hence there is a filtration of W in an increasing sequence of close subspaces $\Phi^{-1}(l_1) \subset \Phi^{-1}(l_2) = H \subset \Phi^{-1}(l_3) \subset \ldots \subset \Phi^{-1}(l_\infty) \subset \mathcal{V}$. It would be interesting to have a characterization of the subspaces of this filtrations in terms of (maybe smoothness of) the continuous functions, however it is not our purpose in this article to go further in this analysis.

3. The Wiener Integral

In this section we turn to the probability aspects of the problem. We shall conclude that the Wiener integral in the AWS which we constructed is exactly the projection of the sequences in $\mathcal{V}$ in the components which corresponds to the components in frequency of the Fourier series of a function $f \in L^2([0,T])$. So, in this sense, the Wiener integral will be a series of independent centered Gaussian variables corresponding to the Fourier coefficients. Firstly we recall the concept of Wiener integral in a general AWS $i : H \to B$. Let $\{e_1, e_2, \ldots\}$ be a complete orthonormal basis of H such that $e_i \in B^*$ (in fact, since $H = L_2$-closure(B^*), approximate any orthonormal basis $\{f_1, f_2, \ldots\}$ for H by elements $\{\widetilde{f_1}, \widetilde{f_2}, \ldots\} \subset B^*$, then get $\{e_1, e_2, \ldots\}$ by Gram-Schmidt orthonormalization). Denote by $P_n : H \to H$ the projection $P_n(x) = \sum_{r=1}^n \langle e_r, x\rangle e_r$ for $x \in H$ and by $\widetilde{P_n} : B \to H$ the linear map given by $\widetilde{P_n}(b) = \sum_{r=1}^n e_r(b)\ e_r$. Note that $\widetilde{P_n} \circ i = P_n$. Given an element $v \in H$ the Wiener integral (with respect to v) is given by

$$\langle v, \cdot\rangle^\sim = L_2\text{-}\lim_{n\to\infty} \left\langle v, \widetilde{P_n}(\cdot)\right\rangle_H .$$

Once $\langle v,\cdot\rangle^{\sim} : B \to \mathbb{R}$ is the L_2-limit of Gaussian random variables, it is also a Gaussian random variable. Note that $||\langle v,\cdot\rangle^{\sim}||_{L_2(B,\mu)} = ||v||_H$, that is the linear map $v \in H \longmapsto \langle v,\cdot\rangle^{\sim} \in L_2(B,\mu)$ is an isometry.

For the well known particular case of the classical Wiener space, given $f \in L^2[0,T]$, the Wiener integral is the Gaussian random variable on $(W,\mathcal{F},\mathbb{W})$ denoted by

$$\int_0^T f(x)\ \mathbb{W}(dx)$$

which is given by the limit in $L_2(W)$ of the white noise integration of simple functions, *i.e.* if $1_{(a,b)}$ is the characteristic function of the interval $(a,b) \subset [0,1]$ then

$$\int_0^T 1_{(a,b)}(x)\ \mathbb{W}(dx) = X(\omega) \tag{3.1}$$

where $X(\omega) = \omega(b) - \omega(a)$. See for instance Nualart (1996) or Ikeda and Watanabe (1984).

Now, consider the isomorphism $\Phi : W \to \mathcal{V} \subset \mathbb{R}^\infty$ established in the last section and let $\mu = \Phi_*(\mathbb{W})$ be the probability measure induced on $\mathcal{V}$. The Wiener integration becomes quite simple in our AWS $i : l_2 \to (\mathcal{V}, ||\cdot||_{\mathcal{V}}, \mu)$. In fact, take the canonical orthonormal basis in l_2 given by $\{e_1 = (1,0,0,\dots), e_2 = (0,1,0,\dots),\dots\}$ such that each element e_i belongs to $\mathcal{V}^*$, the dual space of $\mathcal{V}$. If $(a_n)_{n\geq 1} \in \mathcal{V}$ then $e_i\left((a_n)_{n\geq 1}\right) = a_i$, hence given an element $v = (v_n)_{n\geq 1} \in l_2$, the Wiener integral is simply

$$\langle v,\cdot\rangle^{\sim} = L_2\text{-}\lim_{n\to\infty} \sum_{i=1}^n v_i a_i\,.$$

Similar to the Wiener integral in the space of continuous functions, here the limit is taken necessarily in $L_2(\mathcal{V},\mu)$ once pointwisely it diverges a.s.

Example. The classical Wiener integral in the formula (3.1) is calculated in the space of sequences as follows. Given $f \in L_2([0,T])$, let $v = (v_n)_{n\geq 1} \in l_2$ be the cosine Fourier coefficients of the extended function $\overline{f} \in L_2([0,4T])$ such that $\overline{f}$ is an even function (according to sections 2.1 and 2.2). The sequence v is the image by the isomorphism Φ of the element in the Cameron-Martin space generated by f. The random variable $\int_0^T f\ d\mathbb{W}$ equals the Wiener integral

$$\langle v,\cdot\rangle^{\sim} = L_2\text{-}\lim_{n\to\infty} \sum_{i=1}^n v_i a_i\,.$$

in the sense that $\left(\int_0^T f\ d\mathbb{W}\right)(\omega) = \langle v,\cdot\rangle^{\sim}(\Phi(\omega))$ with equality considered in $L_2(W)$.

A curious class of examples appears when we take, for instance, $f(x) = \sqrt{2T}\cos\left(\frac{\pi n x}{2T}\right)$ for $x \in [0,T]$. The extended even function is $\overline{f}(x) = \sqrt{2T}\cos\left(\frac{\pi n x}{2T}\right)$ for $x \in [0,4T]$. Hence the cosine coefficients in the Fourier series is the element

e_n of the canonical basis in l_2. In this case the Wiener integral is simply the continuous linear functional

$$\langle e_n, \cdot\rangle^{\sim} \left((a_i)_{i\geq 1}\right) = a_n \,.$$

Note that a physical interpretation of the entries of the sequences in $\mathcal{V}$ is that the square of the n-th entry would represent the energy of (the formal) derivative of the trajectories of the Brownian motion contained in the frequency $n/2T$.

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