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ON THE GENERALIZED BOUNDARY VALUE PROBLEM

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Abstract. In the paper it is proved that the generalized linear boundary value problem generates a Fredholm operator. Its index depends on the number of boundary conditions. The existence results of Landesman-Lazer type are given as an application to nonlinear problems by using dual generalized boundary value problems.

The paper deals with the generalized boundary value problem for a nonlinear ordinary differential equation of n-th order.

The properties of mappings generated by such boundary value problems are thoroughly studied by Šeda [9], [10], [11].

An important special case of the generalized boundary value problem is the multipoint boundary value problem. Various existence and multiplicity results for multipoint boundary value problems can be found by Rachůnková [6], [7]. We prove that the linear generalized boundary value problem generates always a linear Fredholm operator. Its index is given by the order of equation and the number of boundary conditions.

A certain type of existence result for two point nonlinear boundary value problems at resonance in L_2 space, given by Grossinho [1], is based on the adjoint boundary value problem to the corresponding linear problem.

If a classical solution is required or boundary conditions are more general an adjoint problem cannot be described as boundary value problem for ordinary differential equation. Instead of the adjoint problem we use a dual boundary value problem and apply the method of Grossinho for the generalized boundary value problem with bounded nonlinearity.

Preliminaries

We consider the generalized boundary value problem

$$(1) \qquad x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)x(t) + f(t, x(t), ..., x^k(t)) = h(t)\,,$$

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$$l_i(x) = 0 \qquad i = 1, \dots, m\,, \tag{2}$$

on the bounded interval $I = [a, b]$, where $a_i \in C(I)$, $h \in C(I)$, $f \in C(I \times R^{k+1})$ and $l_i : C^{n-1}(I) \to R$ are continuous linear functionals.

Boundary conditions (2) are assumed to be linearly independent.

An abstract formulation of the problem (1), (2) is given by the operator equation

$$Lx + Nx = h\,,$$

where

$$L : X \to Z\,,$$

$$L(x(t)) = x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)x(t) \tag{3}$$

is a linear operator,

$$N : X \to Z,$$

$$N(x(t)) = f(t, x(t), ..., x^k(t))$$

is a nonlinear operator and $h \in Z$. The spaces X and Z are real Banach spaces.

Linear problem

In this part we deal with the linear generalized boundary value problem

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)x(t) = h(t)\,, \tag{4}$$

$$l_i(x) = 0 \qquad i = 1, \dots, m\,. \tag{2}$$

We say that the problem (4), (2) *generates* a linear operator $L : X \to Z$ given by (3) where $X = \{x \in C^n(I);\quad l_i(x) = 0, \quad i = 1, \dots, m\}$ and $Z = C(I)$.

We show that the problem (4), (2) generates a Fredholm operator.

Definition [4], [12], [13]**.** Let X, Z be Banach spaces.

A linear operator $L : \operatorname{dom} L \subset X \to Z$ is called Fredholm iff

(i) the null space $N(L)$ is finite dimensional

(ii) the range $R(L)$ is closed and of finite codimension.

Index of L is the number

$$\operatorname{ind} L = \dim N(L) - \operatorname{codim} R(L)\,.$$

A characteristic of Fredholm operators of index zero is given by Nikol'skij [12].

Theorem 1 (Nikol'skij) [12, p.233]. *A linear bounded operator $L : X \to Z$ is Fredholm of index zero iff*

$$L = C + T\,,$$

where C is a linear homeomorphism of X onto Z and $T : X \to Z$ is a linear compact operator.

In case when the order of equation (4) is equal to the number of conditions (2) Nikol'skij theorem follows that the operator L given by (3) is Fredholm of index zero.

Lemma 1 [8, p.56]. *The generalized boundary value problem (4), (2) with $n = m$ generates a Fredholm operator of index 0.*

To prove that the linear operator L defined by the problem (4), (2) is Fredholm of index $k = n - m$ we use two simple generalizations of Nikol'skij theorem.

Theorem 2. *A linear bounded operator $L : X \to Z$ is Fredholm of nonnegative index k iff there is a continuous projection $P : X \to X$ such that $P(X) = X_a$, $dim X_a = k$ and*

$$L = CP' + T\,,$$

where $P' = I - P$ is a continuous projection onto X_b, $X = X_a \oplus X_b$, C is a linear homeomorphism of X_b onto Z and $T : X \to Z$ is a linear compact operator.

Proof. Suppose L is a Fredholm operator of index $k \geq 0$.

Then there is X_a, $\dim X_a = k$ such that $N(L) = X_a \oplus X_1$. As $X = N(L) \oplus X_2$ we denote $X_b = X_1 \oplus X_2$, P a projection onto X_a and $P' = I - P$ a projection onto X_b.

Obviously $L|_{X_b}$ is Fredholm of index zero.

Theorem 1 implies that

$$L|_{X_b} = C + T'$$

where C is a linear homeomorphism of X_b onto Z and $T' : X_b \to Z$ is a linear compact operator.

Now

$$L = LP' = L|_{X_b}P' = (C + T')P' = CP' + T\,.$$

Suppose $L = CP' + T$.

Then $L = (C + T)P' + TP = (C + T')P' + TP$ where $T' = TP'$, $T' : X_b \to Z$ is compact. Theorem 1 implies that $C + T' : X_b \to Z$ is Fredholm of index zero.

As kernel

$$N((C + T')P') = N(C + T') \oplus N(P')$$

and range

$$R((C + T')P') = R(C + T')$$

there is

$$\begin{aligned}\dim N((C + T')P') &= \dim N(C + T') + \dim N(P') = \\ &= \operatorname{codim} R(C + T') + \dim X_a = \operatorname{codim} R(C + T') + k\,.\end{aligned}$$

That means the operator $(C+T')P'$ is Fredholm of index k. Since TP is a compact operator, $L = (C+T')P'+TP$ is a compact perturbation of the operator $(C+T')P'$ with index k and therefore the index of L is again equal to k [13, Theorem 5.E]. $\square$

Theorem 3. *A linear bounded operator $L : X \to Z$ is Fredholm of nonpositive index $-k$ iff there is a finite dimensional space X_1, $\dim X_1 = k$, $X \cap X_1 = \{0\}$ such that*

$$L = C|_X + T\,,$$

where C is a linear homeomorphism of $X \oplus X_1$ onto Z and $T : X \to Z$ is a linear compact operator.

Proof. Let L be Fredholm of index $-k$.

The existence of a finite dimensional space X_1 such that $\dim X_1 = k$, $X \cap X_1 = \{0\}$ is obvious. We define $L_1 : X \oplus X_1 \to Z$ by

$$L_1(x + x_1) = Lx \qquad \text{for } x \in X,\ x_1 \in X_1\,.$$

The operator L_1 is Fredholm of index zero and

$$L_1 = C + T_1$$

where C is a linear homeomorphism of $X \oplus X_1$ onto Z and $T_1 : X \oplus X_1 \to Z$ is a linear compact operator. Then

$$L = L_1|_X = C|_X + T_1|_X = C|_X + T\,.$$

Let $L = C|_X + T$.

We define $T_1 : X \oplus X_1 \to Z$ by

$$T_1(x + x_1) = Tx - Cx_1 \qquad \text{for } x \in X,\ x_1 \in X_1\,.$$

As X_1 is finite dimensional, T_1 is a linear compact operator and $T_1|_X = T$. Now

$$L = C|_X + T_1|_X = (C + T_1)|_X\,.$$

The operator $C + T_1 : X \oplus X_1 \to Z$ is Fredholm of index zero, and

$$(C + T_1)(x + x_1) = (C + T_1)x + (C + T_1)x_1 = (C + T)x \qquad \text{for } x \in X,\ x_1 \in X_1\,.$$

That means $R(C + T_1) = R(L)$ and $N(C + T_1) = X_1 \oplus N(L)$. Now

$$\dim N(C + T_1) = \dim X_1 + \dim N(L) = k + \dim N(L)$$

and

$$\operatorname{codim} R(C + T_1) = \operatorname{codim} R(L)\,.$$

Therefore

$$\operatorname{ind} L = -k\,. \qquad \square$$

Two previous theorems imply the following result for the boundary value problem (4), (2) in case when the order of equation and the number of conditions are different.

Lemma 2. *The generalized boundary value problem* (4), (2) *generates a Fredholm operator of index* $n - m$.

Proof. We denote $X = \{x \in C^n(I); \quad l_i(x) = 0, \quad i = 1, \dots, m\}, \qquad Z = C(I)$.

Case $n \geq m$.

We add $k = n - m$ boundary conditions

$$\tilde{l}_j(x) = 0 \,, \quad j = 1, \dots, k \tag{5}$$

linearly independent on X. There is a set $\{\phi_j, \quad j = 1, \dots, k\} \subset X$ of linearly independent functions such that $\tilde{l}_j(\phi_j) \neq 0$. We denote by $X_a = \text{span}\{\phi_j, \quad j = 1, \dots, k\}$ the k dimensional subspace of X and P a continuous projection onto X_a.

We define a complement of X_a by

$$X_b = \{x \in C^n(I)\,; \quad l_i(x) = 0, \quad i = 1, \dots, m\,; \quad \tilde{l}_j(x) = 0, \quad j = 1, \dots, k\}\,.$$

Lemma 1 implies that the problem (4), (2), (5) generates a Fredholm operator L_1 of index zero defined on X_b.

That means the operator $L : X \to Z$ given by (3) is

$$L = LP + L_1P' = LP + (C + T')P' = CP' + T\,.$$

Theorem 2 implies that $\operatorname{ind} L = k$.

Case $n < m$.

We remove $k = m - n$ linearly independent boundary conditions

$$l_j(x) = 0 \,, \quad j = 1, \dots, k$$

and denote by X_1 the k dimensional space $X_1 = \text{span}\{\phi_j \in C^n(I), \quad j = 1, \dots, k\}$ with basis $\{\phi_j, \ j = 1, \dots, k \quad l_j(\phi_j) \neq 0\}$. Obviously $X \cap X_1 = \{0\}$.

Operator L_1 defined by (3) on $X \oplus X_1$ is Fredholm of index zero. Lemma 1 implies that

$$L_1 = C + T$$

where C is a linear homeomorphism of $X \oplus X_1$ onto Z and $T : X \oplus X_1 \to Z$ is a linear compact operator.

Now

$$L = L|_X$$

and Theorem 3 implies $\operatorname{ind} L = -k$. □

Dual pairs and dual problem

We define a dual problem to the linear problem (4), (2) and prove a Fredholm alternative for dual operators which can be used for nonselfadjoint generalized boundary value problems. Our Theorem 4 below is a modification of Theorem 5.G [13, p.304] for the case of nonselfadjoint boundary conditions.

Definition [13]. Let U, V be Banach spaces and $D : V \times U \to R$ be a bounded bilinear map.

We call $\{V, U\}_D$ a dual pair iff the following separation properties are fulfilled

$$D(v, u) = 0 \qquad \text{for each } u \in U \Rightarrow v = 0\,,$$

$$D(v, u) = 0 \qquad \text{for each } v \in V \Rightarrow u = 0\,.$$

We denote by $\langle v, u\rangle_D = D(v, u)$.

Definition. Let X, Y, Z, W be Banach spaces and $D : Z \times Y \to R$, $\overline{D} : W \times X \to R$ be bounded bilinear maps.

We call the operators $L : X \to Z$ and $L^D : Y \to W$ dual iff for each $x \in X$, and each $y \in Y$

$$\langle Lx, y\rangle_D = \langle L^D y, x\rangle_{\overline{D}}\,.$$

Now we consider the equation

$$Lx = h\,, \qquad L : X \to Z \tag{6}$$

along with the dual equation

$$L^D y = 0\,, \qquad L^D : Y \to W \tag{7}$$

and we formulate the following Fredholm alternative.

Theorem 4. *Let X, Y, Z, W be Banach spaces and $D : Z \times Y \to R$, $\overline{D} : W \times X \to R$ be bounded bilinear maps. We assume that*

(i) $\{Z, Y\}_D$, $\{W, X\}_{\overline{D}}$ are dual pairs,

(ii) $L : X \to Z$ and $L^D : Y \to W$ are dual operators,

(iii) L, L^D are Fredholm with

$$\operatorname{ind} L + \operatorname{ind} L^D = 0\,.$$

Then for each given $h \in Z$ the equation (6) has a solution $x \in X$ iff

$$\langle h, y\rangle_D = 0$$

for each solution $y \in Y$ of the dual equation (7).

Proof. We show that

$$\dim N(L^D) \le \dim N(L^T)$$

where $L^T : Z^* \to X^*$ is the usual adjoint operator to L and Z^*, X^* are the dual spaces to Z, X respectively.

Let $\{y_1, \dots, y_n\}$ be a basis of $N(L^D) \subset Y$. We define linear functionals $f_j \in Z^*$ by

$$f_j(z) = \langle z, y_j\rangle_D \qquad \text{for } z \in Z\,.$$

Then

$$f_j(Lx) = \langle Lx, y_j\rangle_D = \langle L^D y_j, x\rangle_{\overline{D}} = 0$$

or

$$\langle L^T f_j, x\rangle = \langle f_j, Lx\rangle = 0 \quad \text{for each } x \in X\,.$$

Thus $f_j \in N(L^T)$. The separation property of D implies $f_1, \ldots, f_n$ are linearly independent.

Similarly

$$\dim N(L) \le \dim N((L^D)^T)$$

where $(L^D)^T : W^* \to Y^*$.

As

$$\operatorname{codim} R(L) = \dim N((L^T) \qquad \operatorname{codim} R(L^D) = \dim N((L^D)^T)$$

there is

$$\operatorname{ind} L = \dim N(L) - \operatorname{codim} R(L) \le \operatorname{codim} R(L^D) - \dim N((L^D) = -\operatorname{ind} L^D\,.$$

The assumption (iii) implies

$$\dim N(L^D) = \dim N(L^T)\,.$$

Then $N(L^T) = \{f_1, \ldots, f_n\}$, and

$$\langle f_j, h\rangle = 0 \quad \Longleftrightarrow \quad \langle h, y_j\rangle_D = 0\,.$$

The classical Fredholm alternative for the adjoint operator L^T implies the statement of our theorem. □

Example 1. Let us consider the generalized linear boundary value problem

$$x'' + x = h(t)\,, \tag{8}$$

$$x(1) = 0, \qquad \int_0^p x(t)\,dt = 0, \quad 0 < p \le 1\,, \tag{9}$$

on the interval $I = [0, 1]$, with $h \in C(I)$.

We denote

$$X = \{x \in C^2(I)\,; \quad x \text{ satisfies } (9)\},$$

$$Y = \{y \in C^2(I)\,; \quad y(0) = 0\,, \quad \int_{1-p}^1 y(s)\,ds = 0\}\,,$$

$$Z = W = C(I)\,,$$

$$Lx = x'' + x\,.$$

Functionals $D : Z \times Y \to R$ and $\overline{D} : W \times X \to R$ are given by

$$\langle z, y\rangle_D = \int_0^p \int_s^1 z(t)y(t-s)\,dtds\,.$$
$$\langle w, x\rangle_{\overline{D}} = \int_0^p \int_0^{1-s} w(t)x(t+s)\,dtds\,.$$

By a simple computation we obtain

$$\langle Lx, y\rangle_D = \langle Ly, x\rangle_{\overline{D}}\,.$$

Hence the problem

$$y'' + y = 0\,, \tag{10}$$

$$y(0) = 0\,, \qquad \int_{1-p}^1 y(s)\,ds = 0 \tag{11}$$

is a dual boundary value problem to (8), (9).

Lemma 1 implies that both (8), (9) and (10), (11) generate Fredholm operators of index zero.

Theorem 5. *There is a solution of the generalized boundary value problem* (8), (9) *iff*

$$\langle h, y\rangle_D = 0$$

for each solution $y \in Y$ of the dual generalized boundary value problem (10), (11).

Proof. Theorem 4 implies our result. As the assumption (ii) is proved above and the assumption (iii) follows from Lemma 1 we prove that the assumption (i) is fulfilled.

Let $z \in Z$ and

$$\langle z, y\rangle_D = 0 \quad \text{for each } y \in Y\,.$$

There is

$$\langle z, y\rangle_D = \int_0^p \int_s^1 z(t)y(t-s)\,dtds = \int_0^1 z(t)u(t)\,dt\,,$$

where

$$u(t) = \begin{cases} \int_0^t y(t-s)\,ds\,, & \text{for } t \le p\,,\\ \int_0^p y(t-s)\,ds\,, & \text{for } t > p\,. \end{cases}$$

For given t_0 different from 0, p, $1-p$, and 1 we choose $\delta < \frac{p}{2}$ such that $I_\delta = (t_0-\delta, t_0+\delta) \cap \{0, p, 1-p, 1\} = \emptyset$ and a function $y_\delta \in Y$ positive on $(t_0-\delta, t_0)$, negative on $(t_0, t_0+\delta)$, equal to zero otherwise and such that $\int_{I_\delta} y(t)\,dt = 0$. Then the function u_δ is positive on I_δ and equal to zero otherwise. As $\int_0^1 z(t)u_\delta(t)dt = 0$ for each δ sufficiently small, there is $z(t_0) = 0$. Then $z(t) = 0$.

The proof of the second separation property of assumption (i) as well as the proof of separation properties of operator $\overline{D}$ are similar. □

Example 2. We consider the problem

$$x'' + x = h(t)\,, \tag{8}$$

$$\int_0^p x(t)\,dt = 0 \quad \text{where } 0 < p \leq 1\,, \tag{12}$$

on $I = [0, 1]$. We denote $X = \{x \in C^2(I);\ x \text{ satisfies (12)}\}$, $Y = \{y \in C^2(I);\ y(0) = 0,\ \int_{1-p}^1 y(s)\,ds = 0,\ \int_{1-p}^1 y'(s)\,ds = 0\}$, $Z = W = C(I)$, $Lx = x'' + x$. After a similar computation as in the Example 1 and using the same functionals $D, \overline{D}$ we obtain that the problem

$$y'' + y = 0\,, \tag{13}$$

$$y(0) = 0\,, \qquad \int_{1-p}^1 y(s)\,ds = 0\,, \quad \int_{1-p}^1 y'(s)\,ds = 0 \tag{14}$$

is a dual boundary value problem to (8), (12).

Now Lemma 2 implies that the problem (8), (12) generates a Fredholm operator of index 1 and (13), (14) generates a Fredholm operator of index -1.

Theorem 6. *There is a solution of the generalized boundary value problem* (8), (12) *iff*

$$\langle h, y\rangle_D = 0$$

for each solution $y \in Y$ of the dual generalized boundary value problem (13), (14).

The proof is similar to the one of the preceeding theorem.

Application to nonlinear problems

Example 3. We consider the nonlinear generalized boundary value problem

$$x'' + x + f(t, x) = h(t), \tag{15}$$

$$x(2\pi) = 0, \qquad \int_0^{2\pi} x(t)\,dt = 0, \tag{16}$$

on the interval $I = [0, 2\pi]$, where $f : I \times R \to R$ is a continuous and bounded function and $h \in C(I)$.

We formulate an existence result for the generalized boundary value problem (15), (16) by using a condition of Landesman-Lazer type [3].

Theorem 7. *Suppose that*

(i) $|f(t,x)| \le M$ *for each* $x \in R$, $t \in I$,

$$(ii)\ \int_0^{\pi} f_+(t)(1-\cos t)\,dt + \int_{\pi}^{2\pi} f_-(t)(1-\cos t)\,dt > \int_0^{2\pi} h(t)(1-\cos t)\,dt > \int_0^{\pi} f_-(t)(1-\cos t)\,dt + \int_{\pi}^{2\pi} f_+(t)(1-\cos t)\,dt$$

where $f_\pm(t) = \lim\limits_{x\to\pm\infty} f(t,x)$.

Then there is a solution of the problem (15), (16).

Proof. We denote $X = \{x \in C^2(I),\ x \text{ satisfies } (16)\}$, $Z = C(I)$ and $L : X \to Z$, $Lx = x'' + x$. Obviously $N(L) = \mathrm{span}\{\sin t\}$ is a one dimensional subspace of X.

The problem

$$y'' + y = 0\,, \tag{17}$$

$$y(0) = 0\,, \qquad \int_0^{2\pi} y(s)\,ds = 0\,, \tag{18}$$

defines the dual operator $L^D : Y \to W$, where $L^D y = y'' + y = Ly$, $Y = \{y \in C^2(I),\quad y \text{ satisfies } (18)\}$ and $W = Z = C(I)$.

Now $N(L^D) = N(L)$ and Theorem 5 yields

$$R(L) = \{z \in Z, \langle z, \sin t\rangle_D = 0\}\,,$$

with

$$\langle z, \sin t\rangle_D = \int_0^{2\pi} z(t)(1-\cos t)\,dt\,.$$

By the decomposition $X = N(L) \oplus X_2$, $Z = R(L) \oplus Z_2$ the subspaces $N(L)$ and $Z_2 = \mathrm{span}\{(1-\cos t)\}$ are one dimensional and there is an isomorphism $J : Z_2 \to N(L)$, defined by $J(k(1-\cos t)) = k\sin t$. The nonlinear operator N is defined by $N(x)(t) = f(t, x(t))$.

By the standard arguments from [5], [1] the original problem

$$Lx + Nx = h$$

is equivalent to the fixed point problem

$$x = Tx\,,$$

where $T : N(L) \oplus X_2 \to N(L) \oplus X_2$,

$$T(x_1, x_2) = (x_1 - JQ(Nx - h), -L_p^{-1}(I - Q)(Nx - h))\,,$$

$Q : Z \to Z_2$ is a continuous projection onto Z_2 and L_p^{-1} is pseudoinverse operator defined on $R(L)$ which is continuous as operator onto X_2 and compact as operator to Z.

Then the homotopy $H : [0,1] \times Z \to Z$,

$$H(\lambda, x) = (I - \lambda T)x$$

is compact.

We use the Leray-Schauder degree and prove that there is an open bounded set Ω such that

$$\deg(H(\lambda, x), \Omega, 0)$$

is constant and nonzero for each $\lambda \in [0,1]$.

The homotopy invariance of degree is satisfied provided that there is no solution of the equation

$$H(\lambda, x) = 0\,, \tag{19}$$

on $\partial\Omega$.

Let $x = x_1 + x_2$ be a solution of (19). Then

$$x_2 + \lambda L_p^{-1}(I - Q)(Nx - h) = 0$$

and the assumption (i) implies that there is a real constant R_2 such that

$$\|x_2\| < R_2\,.$$

Now (19) is equivalent to

$$Lx + \lambda(Nx - h) = (\lambda - 1)J^{-1}x_1$$

and for each solution y of the dual problem (17), (18) there is

$$\lambda\langle(Nx - h), y\rangle_D = (\lambda - 1)\langle J^{-1}x_1, y\rangle_D\,.$$

Assume now that there are sequences $\lambda_n \in (0,1]$, $x_n \in N(L)$ such that $\|x_n\| \to \infty$ and $H(\lambda_n, x_n + x_{2n}) = 0$, where $\|x_{2n}\| < R_2$. As $x_n = k_n \sin t$ there is $|k_n| \to \infty$. We choose $y = \operatorname{sgn}(k_n) \sin t$. Then

$$\langle(Nx - h), y\rangle_D = \frac{(\lambda - 1)}{\lambda}|k_n|\langle(1 - \cos t), sint\rangle_D \le 0\,.$$

That means

$$\operatorname{sgn}(k_n)\int_0^{2\pi} f(t, k_n \sin t + x_{2n})(1 - \cos t)\,dt \le \operatorname{sgn}(k_n)\int_0^{2\pi} h(t)(1 - \cos t)\,dt\,.$$

The last inequality is in contradiction with (ii).

That means there is a real constant R_1 such that

$$\|x_1\| < R_1$$

for each solution $x = x_1 + x_2$ of (19) and there is no solution of (19) on $[0,1] \times \partial\Omega$ where

$$\Omega = \{x \in C(I)\,,\ x = x_1 + x_2\,,\ \|x_1\| < R_1\,,\ \|x_2\| < R_2\}\,.$$

Then the Leray-Schauder degree

$$\deg(H(1, x), \Omega, 0) = \deg(H(0, x), \Omega, 0) \ne 0$$

and there is a solution $x \in \Omega$ of $H(1, x) = 0$. □

Example 4. Let's consider the same nonlinear differential equation (15) on $I = [0, 2\pi]$, with only one generalized boundary condition

$$\int_0^{2\pi} x(t)\, dt = 0\,. \tag{20}$$

We denote $X = \{x \in C^2(I),\ x \text{ satisfies } (20)\}$, $X_a = \operatorname{span}\{\cos t\}$ and $P : X \to X_a$ a continuous projection onto X_a.

Theorem 8. *Suppose that the assumptions* (i), (ii) *hold.*

Then for each $c \in R$ there is a solution x of the problem (15), (20) *such that $Px = c\cos t$.*

Proof. Again we denote $Z = W = C(I)$ and $L : X \to Z$, $Lx = x'' + x$. Now $N(L) = \operatorname{span}\{\sin t, \cos t\}$ is two dimensional subspace of X. We choose $X_1 = \operatorname{span}\{\sin t\}$, $X_a = \operatorname{span}\{\cos t\}$ and decompose $N(L) = X_a \oplus X_1$.

The problem

$$y'' + y = 0\,, \quad (14)$$

defines the dual operator $L^D : Y \to W$, where $L^D y = y' + y = Ly$, $Y = \{y \in C^2(I),\ \text{y satisfies } (14)\}$.

Now $N(L^D) = \operatorname{span}\{\sin t\}$ and Theorem 6 follows

$$R(L) = \{z \in Z, \langle z, \sin t\rangle_D = 0\}\,.$$

To obtain the statement of our theorem we repeat the proof of Theorem 7 in the space $X_b = X_1 \oplus X_2$ with arbitrary but fixed $x_a \in X_a$. $\square$

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