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GREEN'S $\mathcal{D}$-RELATION FOR THE MULTIPLICATIVE REDUCT OF AN IDEMPOTENT SEMIRING

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Abstract. The idempotent semirings for which Green's $\mathcal{D}$-relation on the multiplicative reduct is a congruence relation form a subvariety of the variety of all idempotent semirings. This variety contains the variety consisting of all the idempotent semirings which do not contain a two-element mono-bisemilattice as a subsemiring. Various characterizations will be given for the idempotent semirings for which the $\mathcal{D}$-relation on the multiplicative reduct is the least lattice congruence.

1. Introduction

We shall assume that the reader is familiar with the basics concerning semigroup theory. We refer to [1] for a general background. Recall that a *band* is a semigroup where every element is an idempotent. Green's $\mathcal{D}$-relation is the least semilattice congruence on a band.

An *idempotent semiring* $(S,+,\cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both $(S,+)$ and $(S,\cdot)$ are bands and such that the distributive laws

$$x(y+z)\approx xy+xz\,,\quad (x+y)z\approx xz+yz \tag{1.1}$$

hold. Thus the idempotent semirings form a variety which will be denoted by $\mathbf{I}$. The intriguing feature about the study of idempotent semirings is the investigation of how the distributive laws (1.1) force the structural characteristics of the two reducts to interact. The present paper is a contribution to this line of investigation.

The variety $\mathbf{Bi}$ of *bisemilattices* is the subvariety of $\mathbf{I}$ determined by the additional identities

$$x+y\approx y+x\,,\quad xy\approx yx\,. \tag{1.2}$$

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Thus, **Bi** consists precisely of the idempotent semirings for which the two reducts are semilattices. *Monobisemilattices* are bisemilattices for which the two reducts are equal semilattices: *monobisemilattices* are bisemilattices which satisfy the identity

$$x + y \approx xy\,. \tag{1.3}$$

The variety of monobisemilattices will be denoted by **M** and the variety of *distributive lattices* by **D**. In [2] it was found that the lattice $L(\mathbf{Bi})$ of subvarieties of **Bi** consists of the five varieties $\mathbf{Bi}, \mathbf{D}, \mathbf{M}, \mathbf{D}\vee\mathbf{M}$ and the trivial variety $\mathbf{T} = \mathbf{D}\cap\mathbf{M}$.

On an idempotent semiring $(S, +, \cdot)$ one may introduce the relations $\leq_+$ and $\leq_\cdot$ by the following: for $a, b \in S$,

$$a \leq_+ b \quad \Leftrightarrow \quad a + b = b = b + a \tag{1.4}$$

$$a \leq_\cdot b \quad \Leftrightarrow \quad ab = a = ba\,. \tag{1.5}$$

The relations $\leq_+$ and $\leq_\cdot$ are partial orders. Thus **D** consists of the bisemilattices for which $\leq_+ = \leq_\cdot$ and **M** consists of the bisemilattices for which $\leq_+ = {}_\cdot\!\geq$. The members of $\mathbf{D} \vee \mathbf{M}$ satisfy the *dual distributive laws*

$$x + yz \approx (x + y)(x + z)\,, \quad xy + z \approx (x + z)(y + z)\,. \tag{1.6}$$

The three element bisemilattice $(S, +, \cdot)$ for which $S = \{a, b, c\}$ with $a \leq_+ b \leq_+ c$ and $a \leq_\cdot c \leq_\cdot b$ does not satisfy these dual distributive laws and thus generates **Bi**. It follows also that $\mathbf{D} \vee \mathbf{M}$ is the subvariety of **Bi** determined by (1.6).

An idempotent semiring is said to be an *idempotent distributive semiring* if it satisfies the dual distributive laws (1.6). The lattice $L(\mathbf{ID})$ of subvarieties of the variety **ID** of idempotent distributive semirings was determined in [3]. It is countably infinite and distributive. According to the above mentioned remarks we have $\mathbf{ID} \cap \mathbf{Bi} = \mathbf{D} \vee \mathbf{M}$.

An identity $u \approx v$ is said to be a *regular identity* if the set of a variables occurring in u coincides with the set of variables occurring in v. A semiring variety contains **M** if and only if it is determined by a set of regular identities. The variety **M** satisfies all regular identities. An identity which is not regular is called a *nonregular identity*. A nonregular identity of the form $x \approx v$, with x a variable, is called a *strongly nonregular identity*.

Theorem 1.1. *For an idempotent semiring S the following are equivalent:*

(i) *S does not contain a two-element monobisemilattice,*

(ii) *S satisfies a nonregular identity,*

(iii) *S satisfies the strongly nonregular identity*

$$x \approx x + xyx + x\,. \tag{1.7}$$

Proof. A two-element monobisemilattice does not satisfy a nonregular identity. Therefore (iii)$\Rightarrow$(ii)$\Rightarrow$(i) holds. Assume that S is an idempotent semiring which does not contain a two-element monobisemilattice and let $a, b \in S$. Then $\{a,\ a +$

$aba + a\}$ forms a monobisemilattice, whence $a = a + aba + a$. Therefore (i)$\Rightarrow$(iii). $\square$

The subvariety of $\mathbf{I}$ determined by (1.7) will be denoted by $\mathbf{N}$.

Corollary 1.2. *For a subvariety* $\mathbf{V}$ *of* $\mathbf{I}$*, either* $\mathbf{M} \subseteq \mathbf{V}$ *or* $\mathbf{V} \subseteq \mathbf{N}$*.*

For a subvariety $\mathbf{V}$ of $\mathbf{I}$ and $S \in \mathbf{I}$ there exists a smallest congruence ρ on S such that $S/\rho \in \mathbf{V}$. This congruence will be called the *least* $\mathbf{V}$*-congruence* on S.

Corollary 1.3. *For any* $S \in \mathbf{I}$*, the congruence relation on* S *generated by* $\leq_+ \cap {}_{\cdot}\!\geq$ *is the least* $\mathbf{N}$*-congruence on* S*.*

Proof. Let ν be the least $\mathbf{N}$-congruence on S. If $a, b \in S$ and $a \ (\leq_+ \cap {}_{\cdot}\!\geq)\ b$, then $\{a\nu, b\nu\}$ form a monobisemilattice, and since S/ν does not contain nontrivial monobisemilattices, we have $a\nu = b\nu$. Therefore ν contains the congruence generated by $\leq_+ \cap {}_{\cdot}\!\geq$.

Let ρ be a congruence on S which does not contain ν, that is, $S/\rho \notin \mathbf{N}$. By Theorem 1.1 there exist $a, b \in S$ such that

$$a\rho \neq (a + aba + a)\rho$$

in S/ρ. Since $a \ (\leq_+ \cap {}_{\cdot}\!\geq)\ a + aba + a$ this entails that ρ does not contain the congruence generated by $\leq_+ \cap {}_{\cdot}\!\geq$. It follows that the congruence generated by $\leq_+ \cap {}_{\cdot}\!\geq$ contains ν. $\square$

For an idempotent semiring $(S, +, \cdot)$ we denote Green's $\mathcal{D}$-relation on the additive [multiplicative] reduct by $\overset{+}{\mathcal{D}}$ $[\dot{\mathcal{D}}]$. For $a \in S$, we denote the $\overset{+}{\mathcal{D}}$-class $[\dot{\mathcal{D}}$-class$]$ containing a by $\overset{+}{D}_a$ $[\dot{D}_a]$. Similarly, corresponding to Green's $\mathcal{L}$- and $\mathcal{R}$-relation we have the relations $\overset{+}{\mathcal{L}}, \dot{\mathcal{L}}$ and $\overset{+}{\mathcal{R}}, \dot{\mathcal{R}}$, respectively. Recall that $\overset{+}{\mathcal{D}}$ $[\dot{\mathcal{D}}]$ is the least semilattice congruence on the band $(S, +)$ $[(S, \cdot)]$.

Theorem 1.4. *For an idempotent semiring* $(S, +, \cdot)$*,* $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}} = \overset{+}{\mathcal{D}}\dot{\mathcal{D}}\overset{+}{\mathcal{D}}$ *and the congruence relation generated by* $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}}$ *is the least* $\mathbf{Bi}$*-congruence. The congruence generated by* $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}} \vee (\leq_+ \cap {}_{\cdot}\!\geq)$ *is the least* $\mathbf{D}$*-congruence.*

Proof. Assume that $a, b, c, d \in S$ such that $a \,\dot{\mathcal{D}}\, c \,\overset{+}{\mathcal{D}}\, d \,\dot{\mathcal{D}}\, b$. From $c = c + d + c$ and $a = aca$ we deduce that $a = aca = aca + ada + aca = a + ada + a$. From $d = d + c + d$ and $a = aca$ we find that $ada = ada + aca + ada = ada + a + ada$. We found that $a \,\overset{+}{\mathcal{D}}\, ada$, and similarly we can prove that $b \,\overset{+}{\mathcal{D}}\, bcb$. Since $\dot{\mathcal{D}}$ is the least semilattice congruence on $(S, \cdot)$ we have from $a \,\dot{\mathcal{D}}\, c$ and $d \,\dot{\mathcal{D}}\, b$ that $ada \,\dot{\mathcal{D}}\, bcb$. Therefore $a \,\overset{+}{\mathcal{D}}\, ada \,\dot{\mathcal{D}}\, bcb \,\overset{+}{\mathcal{D}}\, b$. We proved that $\dot{\mathcal{D}}\overset{+}{\mathcal{D}}\dot{\mathcal{D}} \subseteq \overset{+}{\mathcal{D}}\dot{\mathcal{D}}\overset{+}{\mathcal{D}}$ and consequently that $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}} = \overset{+}{\mathcal{D}}\dot{\mathcal{D}}\overset{+}{\mathcal{D}}$.

The least $\mathbf{Bi}$-congruence on S contains the least semilattice congruence $\overset{+}{\mathcal{D}}$ on $(S, +)$ and the least semilattice congruence $\dot{\mathcal{D}}$ on $(S, \cdot)$. Clearly each congruence on the semiring S containing both $\overset{+}{\mathcal{D}}$ and $\dot{\mathcal{D}}$ is a **Bi**-congruence. Therefore the congruence generated by $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}}$ is the least **Bi**-congruence. From Corollary 1.3 and since $\mathbf{D} \subseteq \mathbf{N}$ we now have that the least $\mathbf{D}$-congruence contains $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}} \vee (\leq_+ \cap \,_{\cdot}\geq)$. We have that $\mathbf{Bi} \cap \mathbf{N} = \mathbf{D}$ and therefore by Corollary 1.3 and the above we have that the congruence generated by $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}} \vee (\leq_+ \cap \,_{\cdot}\geq)$ contains the least $\mathbf{D}$-congruence. ☐

We include the following result for later reference.

Theorem 1.5. *Let $(S, +, \cdot)$ be an idempotent semiring. Then $\overset{+}{\mathcal{D}}$ is a congruence relation on the semiring $(S, +, \cdot)$, while $\overset{+}{\mathcal{L}}$ and $\overset{+}{\mathcal{R}}$ are congruence relations on the band $(S, \cdot)$. The $\overset{+}{\mathcal{L}}$- and $\overset{+}{\mathcal{R}}$-classes form subsemirings.*

The *dual* of a statement (P) is the statement which results from (P) by interchanging the role of $\cdot$ and $+$. It should be noted that the dual of a statement which holds true for all idempotent semirings need not be true. Indeed, while $\overset{+}{\mathcal{D}}$ is a congruence relation on any idempotent semiring, $\dot{\mathcal{D}}$ need not be, as can be seen from the following

Example 1.6. Let $(S, +, \cdot)$ have the following addition and multiplication table

$+$	a	b	c	w
a	a	b	c	w
b	b	b	c	c
c	c	c	c	c
w	w	c	c	w

$\cdot$	a	b	c	w
a	a	a	a	a
b	b	b	b	b
c	b	b	c	c
w	a	b	c	w

One verifies that $(S, +)$ is a semilattice, and $(S, \cdot)$ a band with $\dot{\mathcal{L}} = \dot{\mathcal{D}}$. Further, the distributive laws (1.1) hold and so $(S, +, \cdot)$ is an idempotent semiring. However, though $a \,\dot{\mathcal{L}}\, b$ we have that $a + w = w$ and $b + w = c$ are not $\dot{\mathcal{D}}$-related, and thus $\dot{\mathcal{D}} = \dot{\mathcal{L}}$ is not a congruence. We note that $\{w, c\}$ forms a two-element-monobisemilattice, and so $S \notin \mathbf{N}$. Nevertheless the greatest bisemilattice homomorphic image of S is a two-element distributive lattice and thus belongs to $\mathbf{N}$.

From the following we see that the idempotent semirings whose $\dot{\mathcal{D}}$-relation is a congruence form a subvariety of $\mathbf{I}$. This variety is necessarily a proper subvariety of $\mathbf{I}$ because of Example 1.6.

Theorem 1.7. *Let S be an idempotent semiring. Then the $\dot{\mathcal{D}}$-relation on S is a congruence if and only if S satisfies the identities*

$$\begin{aligned} (xy+z)(yx+z)(xy+z) &\approx xy+z\,, \\ (x+yz)(x+zy)(x+yz) &\approx x+yz\,. \end{aligned} \tag{1.8}$$

Proof. Assume that $\dot{\mathcal{D}}$ is a congruence on S and let $a, b, c \in S$. Then $ab \,\dot{\mathcal{D}}\, ba$ and also $ab + c \,\dot{\mathcal{D}}\, ba + c$, whence $(ab+c)(ba+c)(ab+c) = ab+c$, and in a left-right dual way for the addition one show that $(a + bc)(a + cb)(a + bc) = a + bc$. Therefore the identities (1.8) are satisfied in S.

Conversely, assume that the idempotent semiring S satisfies the identities (1.8). Let $a \dot{\mathcal{D}} b$, thus $a = (ab)(ba)$ and $b = (ba)(ab)$. Let c be any element of S. Substituting x by ab, y by ba and z by c in the first identity of the statement we obtain $(a + c)(b + c)(a + c) = a + c$. By symmetry also $(b + c)(a + c)(b + c) = b + c$ holds and thus $a + c \,\dot{\mathcal{D}}\, b + c$. From the second identity one obtains similarly that $c + a \,\dot{\mathcal{D}}\, c + b$. Since $\dot{\mathcal{D}}$ is also a congruence on $(S, \cdot)$ we may thus conclude that $\dot{\mathcal{D}}$ is a congruence on S. □

Corollary 1.8. *$\dot{\mathcal{D}}$ is not a congruence relation on the free idempotent semiring generated by three free generators.* □

We remark that the idempotent semiring of Example 1.6 is generated by $\{a, c, w\}$ but not by any two of its elements. This leaves the question open whether $\dot{\mathcal{D}}$ is a congruence on the free idempotent semiring generated by two free generators.

We still have a weak analogue of Theorem 1.5.

Theorem 1.9. *Let S be an idempotent semiring. Then the $\dot{\mathcal{L}}$-, $\dot{\mathcal{R}}$- and $\dot{\mathcal{D}}$-classes form subsemirings of S.*

Proof. If $a \dot{\mathcal{L}} b$ in S then $ab = a$, $ba = b$ and so $a(a + b) = a + ab = a + a = a$, $(a + b)a = a + ba = a + b$, and therefore $a \,\dot{\mathcal{L}}\, a + b$. From this follows that the $\dot{\mathcal{L}}$-classes form subsemirings. The left-right dual for the multiplication yields that the $\dot{\mathcal{R}}$-classes form subsemirings.

Assume that $a \dot{\mathcal{D}} b$ in S. Then by the foregoing $a \dot{\mathcal{R}} ab \dot{\mathcal{R}} ab+a$ and $a \dot{\mathcal{L}} ba \dot{\mathcal{L}} ba+a$. Hence

$$\begin{aligned} a \dot{\mathcal{D}} (ba + a)(ab + a) &= bab + ba + ab + a \\ &= b + ba + ab + a = (b + a)^2 = b + a\,. \end{aligned}$$

We proved that the $\dot{\mathcal{D}}$-classes form subsemirings. □

A set of identities is *self-dual* if for any identity in the set, interchanging the role of $+$ and $\cdot$ yields an identity which is again in the set. Clearly if a variety can be given by a self-dual set of identities, and (P) is a statement which holds true for all the members of this variety, then its dual also holds true for all the members of the variety. The variety **ID** of idempotent distributive semirings can be given by a self-dual set of identities, and so $\dot{\mathcal{D}}$ is a congruence on any $S \in$ **ID**, since $\overset{+}{\mathcal{D}}$ is:

Theorem 1.10. *$\dot{\mathcal{D}}$ is a congruence on an idempotent distributive semiring.*

Corollary 1.11. *$\dot{\mathcal{D}}$ is a congruence on any idempotent semiring whose additive or multiplicative reduct is a rectangular band.*

Proof. An idempotent semiring whose additive or multiplicative reduct is a rectangular band belongs to **ID** [3]. □

The main objective of Section 2 will be to show that $\dot{\mathcal{D}}$ is a congruence relation on every member of **N**. In other words, we show that **N** is a subvariety of the variety of idempotent semirings determined by (1.8). Using Theorem 1.4, this will allow to represent the least lattice congruence on the members of **N** in a much simpler form.

Of course **N** contains semirings which are not in **ID**:

Example 1.12. Let $(S, +, \cdot)$ be such that $S = \{a, b, o\}$ with the following addition and multiplication table:

+	a	b	o
a	a	b	a
b	b	b	b
o	a	b	o

·	a	b	o
a	a	a	o
b	b	b	o
o	o	o	o

Thus $(S, +, \cdot)$ is an idempotent semiring which belongs to **N**. This idempotent semiring has an additive reduct which is a semilattice. The dual distributive laws (1.6) do not hold since $bo + a = a$ whereas $(b + a)(o + a) = b$. $\dot{\mathcal{D}} = \dot{\mathcal{L}}$ is the least lattice congruence on S.

2. The $\dot{\mathcal{D}}$-relation on the members of N

We set out to prove that $\dot{\mathcal{D}}$ is a congruence relation on every member of **N**, that is according to Theorem 1.1, on every idempotent semiring which does not contain a two-element monobisemilattice. This in particular explains the presence of a two-element monobisemilattice in the counterexample 1.6. Our result then entails in view of Corollary 1.3 and Theorem 1.4 that $\overset{+}{\mathcal{D}} \vee \dot{\mathcal{D}} = \overset{+}{\mathcal{D}}\dot{\mathcal{D}}\overset{+}{\mathcal{D}}$ is the least lattice congruence on any member of **N**. We conclude the section with the investigation of the special situation where $\dot{\mathcal{D}}$ itself is the least lattice congruence.

We proceed through a sequence of lemmas and draw our conclusions at the end.

Lemma 2.1. *Let $S \in \mathbf{I}$. If $a, b, w \in S$ such that $a(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{L}})b$ then $a+w\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{L}})\ b+w$.*

Proof. Clearly $a \overset{+}{\mathcal{L}} b$ implies that $a + w \overset{+}{\mathcal{L}} b + w$.

We shall assume that $a(\dot{\mathcal{L}} \cap \overset{+}{\mathcal{L}})\, b$ and we put $u = a + w + a$ and $v = b + w + b$. Then

$$va = (b + v)a = b + va\,,$$

whence

$$va = vb + va = v(b + a) = vb\,.$$

Again, from $b + v = v$ we then have

$$v = vb + v = va + v = v(a + v)\,.$$

Thus, $v = v(a+v)$ entails $v\,\dot{\mathcal{L}}\,(a+v)v$, whereas $v = b+w+b+a = v+a$ entails $v\,\overset{+}{\mathcal{L}}\,a+v$. Since by Theorem 1.5 the $\overset{+}{\mathcal{L}}$-classes form subsemirings, this gives $v\,(\dot{\mathcal{L}}\cap\overset{+}{\mathcal{L}})\,(a+v)v$.

We found that $v = v(a+v)$. Interchanging the role of a and b we find similarly that $u = u(b+u)$. Since $a\,\overset{+}{\mathcal{L}}\,b$, the elements $u, v, a+v, b+u$ and $(a+v)v$ belong to the same $\overset{+}{\mathcal{D}}$-class. By Theorem 1.5 the $\overset{+}{\mathcal{D}}$-class $\overset{+}{D}_u$ forms a subsemiring and by Corollary 1.11 $\dot{\mathcal{D}}$ is a congruence on $\overset{+}{D}_u$. Further, $\overset{+}{D}_u$ satisfies the dual distributive laws (1.6). Indeed, if $a, b, c \in \overset{+}{D}_u$, then

$$\begin{aligned}(a+b)(a+c) &= (a+b)a+(a+b)c\\ &= a+ba+ac+bc = a+bc\end{aligned}$$

since $a, ba, ac, bc \in \overset{+}{D}_u$. Therefore by (1.6)

$$\begin{aligned}(a+v)v+(b+u) &= (a+v+b+u)(v+b+u)\\ &= (a+b+w+b+a+w+a)(b+w+b+a+w+a)\\ &= (a+w+b+w+a)(b+w+a)\\ &= (a+w+a)(b+w+a) = u(b+u) = u = a+w+a \qquad (2.1)\end{aligned}$$

where $a+w+b+w = a+w$ follows from $a\,\overset{+}{\mathcal{L}}\,b$.

Further,

$$v+(b+u) = b+w+b+a+w+a = b+w+a\,. \qquad (2.2)$$

Since $\dot{\mathcal{D}}$ is a congruence on $\overset{+}{D}_u$, we have from $v\,\dot{\mathcal{L}}\,(a+v)v$ that

$$(a+v)v+(b+u) \quad \dot{\mathcal{D}} \quad v+(b+u)\,. \qquad (2.3)$$

The equalities (2.1), (2.2) and (2.3) then yield

$$a+w+a\,\dot{\mathcal{D}}\,b+w+a\,. \qquad (2.4)$$

Again, since $a+w \in \overset{+}{D}_u$ and $\dot{\mathcal{D}}$ is a congruence on $\overset{+}{D}_u$, we have from (2.4) that

$$\begin{aligned}a+w &= (a+w+a)+(a+w)\\ &\;\dot{\mathcal{D}}\;(b+w+a)+(a+w)\\ &= b+w \qquad\qquad (\text{since } a\,\overset{+}{\mathcal{L}}\,b),\end{aligned}$$

as required.

We proved that if $a(\dot{\mathcal{L}}\cap\overset{+}{\mathcal{L}})\,b$, then $a+w(\dot{\mathcal{D}}\cap\overset{+}{\mathcal{L}})\,b+w$. Applying left-right duality for the multiplication we can show that if $a\,(\dot{\mathcal{R}}\cap\overset{+}{\mathcal{L}})\,b$ then $a+w\,(\dot{\mathcal{D}}\cap\overset{+}{\mathcal{L}})\,b+w$. Thus, if instead we have $a(\dot{\mathcal{D}}\cap\overset{+}{\mathcal{L}})\,b$, then by Theorem 1.5,

$$a\,(\dot{\mathcal{L}}\cap\overset{+}{\mathcal{L}})\,ba\,(\dot{\mathcal{R}}\cap\overset{+}{\mathcal{L}})\,b$$

and so by the above

$$a + w\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{L}})\ ba + w\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{L}})\ b + w\,. \qquad \square$$

Applying left-right duality for the addition we obtain similarly

Lemma 2.2. *Let $S \in \mathbf{I}$. If $a, b, w \in S$ such that $a\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\ b$, then $w + a\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\ w + b$.*

Lemma 2.3. *Let $S \in \mathbf{N}$. If $c, d, w \in S$ such that $c(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, d$ and $w + c = c$, then $c + w\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, d + w$.*

Proof. From $w + c = c$ and $c\ \overset{+}{\mathcal{R}}\ d$ we also have $w + d = d$. Clearly $c + w\, \overset{+}{\mathcal{R}}\, c\, \overset{+}{\mathcal{R}}\, d\, \overset{+}{\mathcal{R}}\, d + w$.

We first assume that $c\, \dot{\mathcal{R}}\, d$. Applying (1.7) and $w + c = c$ we obtain

$$\begin{aligned} w &= w + wcw + w = w(w + c + w)w \\ &= w(c + w)w = w(cw + w) \end{aligned}$$

and since also $(cw+w)w = cw+w$, we have that $w\ \dot{\mathcal{L}}\ cw+w$. Similarly $w\ \dot{\mathcal{L}}\ dw+w$ and thus $cw + w\ \dot{\mathcal{L}}\ dw + w$. Also

$$\begin{aligned} (cw + w) + (dw + w) &= cw + (w + d)w + w = cw + dw + w \\ &= (c + d)w + w = dw + w\,, \end{aligned}$$

and similarly

$$(dw + w) + (cw + w) = cw + w\,.$$

Therefore $cw + w\, (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, dw + w$, and so by Lemma 2.2,

$$d + wd + cw + w\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\ d + wd + dw + w\,,$$

that is,

$$(c + w)(d + w)\ \dot{\mathcal{D}}\ (d + w)^2 = d + w\,.$$

By symmetry we also have

$$(d + w)(c + w)\ \dot{\mathcal{D}}\ c + w$$

and therefore $c + w\, \dot{\mathcal{D}}\ d + w$.

Using left-right duality for the multiplication one shows that if $c(\dot{\mathcal{L}} \cap \overset{+}{\mathcal{R}})\, d$ and $w + c = c$, then again $c + w\, (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, d + w$. Therefore, if instead $c\ (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, d$ and $w + c = c$, then in view of Theorem 1.5 we have

$$c\ (\dot{\mathcal{R}} \cap \overset{+}{\mathcal{R}})\ cd\ (\dot{\mathcal{L}} \cap \overset{+}{\mathcal{R}})\ d$$

with $w + cd = cd$, and so by the above

$$c + w\, (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, cd + w\, (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, d + w\,. \qquad \square$$

Lemma 2.4. *Let $S \in \mathbf{N}$. If $a, b, w \in S$ such that $a(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, b$, then $w + a + w\, (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, w + b + w$.*

Proof. From Lemma 2.2 we have that $w+a\, (\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, w+b$. If we put $c = w+a,\ d = w+b$, then the desired result follows from Lemma 2.3. ☐

Lemma 2.5. *Let $S \in \mathbf{N}$. If $a, b, u \in S$ such that $a\, \dot{\mathcal{L}}\, b,\ a+b = b = b+a,\ a+u = u = u+a$, then $u+b\, \dot{\mathcal{L}}\, u\, \dot{\mathcal{L}}\, b+u$.*

Proof. We shall first assume that $a\, \dot{\mathcal{L}}\, b$. From $u = a+u+a$ and since S satisfies the identity (1.7) we have that $aua = a + aua + a = a$. Since $a\, \dot{\mathcal{L}}\, b$ this also implies that $b = bub$, whence $a\, \dot{\mathcal{L}}\, b\, \dot{\mathcal{L}}\, ua\, \dot{\mathcal{L}}\, ub$ and therefore also $uau\, \dot{\mathcal{L}}\, ubu$. From $a+u = u = u+a$ we have $uau + u = u = u + uau$ and from $a+b = b = b+a$ we have $uau + ubu = ubu = ubu + uau$. Therefore

$$\begin{aligned} ubu + u + ubu &= uau + (ubu)(uau) + u + ubu \\ &= (u+ubu)(uau) + u + ubu = (u+ubu)(uau + u + ubu) \\ &= (u+ubu)^2 = u + ubu\,. \end{aligned}$$

Applying left-right duality for the addition, we thus have

$$u + ubu = ubu + u + ubu = ubu + u. \tag{2.5}$$

Since S satisfies the identity (1.7) we have from (2.5)

$$u = u + ubu + u = (u+ubu) + (ubu+u) = u + ubu = ubu + u\,.$$

Therefore

$$\begin{aligned} uau = uuau &= (ubu+u)\, uau = ubu\, uau + uau \\ &= ubu + uau \qquad (\text{since } uau\, \dot{\mathcal{L}}\, ubu) \\ &= ubu\,. \end{aligned}$$

Since $uau\, \dot{\mathcal{R}}\, ua\, \dot{\mathcal{L}}\, ub\, \dot{\mathcal{R}}\, ubu$, this implies that $ua = ub$. Therefore

$$\begin{aligned} (b+u)(a+u) &= ba + ua + bu + u = b + ub + bu + u \\ &= (b+u)^2 = b + u \end{aligned}$$

and similarly $(a+u)(b+u) = a+u$. We proved that $u = a+u\ \dot{\mathcal{L}}\ b+u$. Applying left-right duality for the addition we find that $u = u+a\, \dot{\mathcal{L}}\, u+b$, whence $u+b\, \dot{\mathcal{L}}\, u\, \dot{\mathcal{L}}\, b+u$. ☐

Lemma 2.6. *Let $S \in \mathbf{N}$. If $p, q, v \in S$ such that $p\, \dot{\mathcal{L}}\, q,\ q+p = p = p+q,\ v+q = q,\ q+v = v$, then $p+v\, \dot{\mathcal{L}}\, q+v$.*

Proof. Clearly $v+p = v+q+p = q+p = p$. Since S satisfies the identity (1.7) we have

$$v = v + v(p+v)v + v = v(v+p+v)v = v(p+v)v$$

that is,

$$q + v = (q+v)(p+v)(q+v)\,. \tag{2.6}$$

From $q+v=v$ and $v+p=p$ we have $vq+v=v$ and $v+pv=pv$, and thus

$$\begin{aligned}(p+v)(q+v) &= p+pv+vq+v = p+pv+v \\ &= p+v+pv+v = (p+v)+(p+v)v \\ &= (p+v)(p+v+v) = (p+v)^2 = p+v\,. \end{aligned} \tag{2.7}$$

From (2.6) and (2.7) then follows that $p+v\,\dot{\mathcal{L}}\,q+v$. □

Lemma 2.7. *Let $S \in \mathbf{N}$. If $a, b, w \in S$ such that $a\,\dot{\mathcal{D}}\,b$ and $a+b=b=b+a$, then $a+w\,\dot{\mathcal{D}}\,b+w$ and $w+a\,\dot{\mathcal{D}}\,w+b$.*

Proof. We shall first assume that $a\,\dot{\mathcal{L}}\,b$. We put $u=a+w+a$ and we infer from Lemma 2.5 that $u+b\ \dot{\mathcal{L}}\ u\ \dot{\mathcal{L}}\ b+u$, that is,

$$(a+w+b)\ \dot{\mathcal{L}}\ (a+w+a)\ \dot{\mathcal{L}}\ (b+w+a)\,. \tag{2.8}$$

From Theorem 1.9 we then have that $a+w+a$ is $\dot{\mathcal{L}}$-related to $(a+w+b)+(b+w+a)=a+w+b+w+a$. Putting $p=a+w+b+w+a$, $q=a+w+a$, $v=a+w$, we have that $p\ \dot{\mathcal{L}}\ q$, $q+p=p+q=p$, $v+q=q$ and $q+v=v$, and so according to Lemma 2.6, $p+v\,\dot{\mathcal{L}}\,q+v$, that is,

$$a+w+b+w+a+w\ \dot{\mathcal{L}}\ a+w\,. \tag{2.9}$$

The elements $b+w+a$, $a+w+b$ and $b+w+a+w$ are $\overset{+}{\mathcal{D}}$-related. By Theorem 1.5 and Corollary 1.11 this $\overset{+}{\mathcal{D}}$-class forms an idempotent semiring for which $\dot{\mathcal{D}}$ is a congruence, hence from (2.8) we obtain

$$(b+w+a)+(b+w+a+w)\ \dot{\mathcal{D}}\ (a+w+b)+(b+w+a+w)$$

that is,

$$(b+w+a+w)\ \dot{\mathcal{D}}\ (a+w+b+w+a+w)\,. \tag{2.10}$$

We have $(b+w+a)+(a+w+b)=b+w+b$ since $a+b=b$. From (2.8) and Theorem 1.9 we have that $(b+w+b)\ \dot{\mathcal{L}}\ (b+w+a)$ and evidently also $(b+w+b)\ \overset{+}{\mathcal{R}}\ (b+w+a)$ in the $\overset{+}{\mathcal{D}}$-class containing $b+w$. Putting $c=b+w+b$ and $d=b+w+a$, we have $c(\dot{\mathcal{D}}\cap\overset{+}{\mathcal{R}})\,d$ and so according to Lemma 2.4, $w+c+w\ (\dot{\mathcal{D}}\cap\overset{+}{\mathcal{R}})\ w+d+w$. Thus, in our former notation,

$$w+(b+w+b)+w\ \dot{\mathcal{D}}\ w+(b+w+a)+w\,,$$

that is,

$$w+b+w\ \dot{\mathcal{D}}\ w+b+w+a+w\,. \tag{2.11}$$

Since the two elements mentioned in (2.11) belong to the $\overset{+}{\mathcal{D}}$-class of $b+w$ and since $\dot{\mathcal{D}}$ is a congruence for this idempotent semiring $\overset{+}{D}_{b+w}$, we have from (2.11) that

$$(b+w)+(w+b+w)\ \dot{\mathcal{D}}\ (b+w)+(w+b+w+a+w)$$

that is,

$$(b+w)\ \dot{\mathcal{D}}\ (b+w+a+w)\,. \tag{2.12}$$

From (2.9), (2.10) and (2.12) we thus find that $a+w\,\dot{\mathcal{D}}\,b+w$.

If $a\ \dot{\mathcal{R}}\ b$ and $a+b=b=b+a$, then applying left-right duality for the multiplication we find $a+w\ \dot{\mathcal{D}}\ b+w$. We now assume that $a\ \dot{\mathcal{D}}\ b$ and thus $a\,\dot{\mathcal{L}}\,ba\,\dot{\mathcal{R}}\,b$. If $a+b=b=b+a$, then $a+ba=ba=ba+a$ and $ba+b=b=b+ba$, hence from the foregoing we also have

$$a+w\ \dot{\mathcal{D}}\ ba+w\ \dot{\mathcal{D}}\ b+w\,.$$

We proved that if $a\ \dot{\mathcal{D}}\ b$ and $a+b=b=b+a$, then $a+w\dot{\mathcal{D}}\,b+w$. Applying left-right duality for the addition we find similarly that the same premise entails $w+a\,\dot{\mathcal{D}}\,w+b$. $\square$

Lemma 2.8. *Let $S\in\mathbf{N}$ and $u,b\in S$ such that $u+b=u$. Then $(b+bu+v)\dot{\mathcal{D}}\ b+v\ \dot{\mathcal{D}}\ (b+ub+v)$ for all $v\in S$.*

Proof. Since S satisfies the identity (1.7) we have that

$$b=b+bub+b=b(b+u+b)b=b(b+u)b=(b+bu)b$$

and therefore $b+bu\ \dot{\mathcal{R}}\,b$. Also $bu+b=b(u+b)=bu$. Thus, putting $a=b+bu$, we have $a\ \dot{\mathcal{R}}\,b$, $a+b=a=b+a$, and so from Lemma 2.7 it follows that $a+v\ \dot{\mathcal{D}}\ b+v$ for all $v\in S$. We proved that $b+bu+v\,\dot{\mathcal{D}}\,b+v$ for all $v\in S$. Applying left-right duality for the multiplication, we find similarly that $b+ub+v\,\dot{\mathcal{D}}\,b+v$ for all $v\in S$. $\square$

Lemma 2.9. *Let $S\in\mathbf{N}$. If $p,q,v\in S$ such that $p\ (\overset{+}{\mathcal{R}}\cap\dot{\mathcal{D}})q$, $pv=p$, $qv=q$ and $v+q=v$. Then $p+v\ \dot{\mathcal{L}}\ q+v$.*

Proof. We have that $vp=vpv\ \dot{\mathcal{L}}\ p\ \dot{\mathcal{D}}\ q\ \dot{\mathcal{L}}\ vq=vqv$. From $v+q=v$ we see that $v+vq=v$. Since $p\ \overset{+}{\mathcal{R}}\ q$ we also have that $vp\ \overset{+}{\mathcal{R}}\ vq$ and from Theorem 1.5 we see that also $vpq=vp\ vq$ and $vqp=vq\ vp$ belong to the $\overset{+}{\mathcal{R}}$-class of vp and vq. We have

$$\begin{aligned} vqp &= vqpv = vqp(v+vq) = vqpv+vqpvq \\ &= vqp+vq = vq \end{aligned}$$

since $vqp\ \overset{+}{\mathcal{R}}\ vq$. Similarly

$$vpq=(v+vq)pq=vpq+vqpq=vpq+vq=vq$$

since $vpq\,\overset{+}{\mathcal{R}}\,vq$. Consequently $vq=(vpq)(vqp)=vp$.

Now

$$\begin{aligned} (p+v)(vp+v) &= pvp+pv+vp+v=(p+v)^2=p+v\,, \\ (vp+v)(p+v) &= vp+vpv+vvp+v=(vp+v)^2=vp+v \end{aligned}$$

and so $p+v\ \dot{\mathcal{L}}\ vp+v$. Similarly, $q+v\ \dot{\mathcal{L}}\ vq+v$. However, since $vp=vq$ we have $p+v\ \dot{\mathcal{L}}\ q+v$, as required. $\square$

Lemma 2.10. *Let $S \in \mathbf{N}$. If $a, b, w \in S$ such that $a(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})b$, then $a + w \,\dot{\mathcal{D}}\, b + w$.*

Proof. We shall first assume that $a(\dot{\mathcal{R}} \cap \overset{+}{\mathcal{R}})b$. By Lemma 2.4 we have that

$$w + a + w \,\dot{\mathcal{D}}\, w + b + w \tag{2.13}$$

and by Lemma 2.2 that

$$w + a \;(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}}) w + b\,. \tag{2.14}$$

Putting $u = w + b$, we have that $u = u + b$ and so by Lemma 2.8, for all $t \in S$,

$$b + t \,\dot{\mathcal{D}}\, b + bu + t \,\dot{\mathcal{D}}\, b + ub + t\,.$$

In particular then

$$b + bu + u \,\dot{\mathcal{D}}\, b + u \tag{2.15}$$

and

$$b + au + u \,\dot{\mathcal{D}}\, b + ub + au + u = (a + u)(b + u)\,. \tag{2.16}$$

Put $p = b + au, q = b + bu$ and $v = b + u$. Since $a(\dot{\mathcal{R}} \cap \overset{+}{\mathcal{R}})b$ we also have $au(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, bu$, and from Lemma 2.2 then $b + au \;(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, b + bu$, that is, $p \;(\dot{\mathcal{D}} \cap \overset{+}{\mathcal{R}})\, q$. Also $p = b + au = a(b + u) = av$, $q = b + bu = b(b + u) = bv$ and so $pv = p$, $qv = q$. Further $b + u + b + bu = b + u + bu = b + (u + b)u = b + u$ since $u + b = u$, whence $v + q = v$. By Lemma 2.9 we have $p + v \,\dot{\mathcal{L}}\, q + v$, that is,

$$b + au + b + u \,\dot{\mathcal{D}}\, b + bu + b + u\,. \tag{2.17}$$

Since

$$au + b = a(u + b) = au, \;\; bu + b = b(u + b) = bu\,,$$

(2.17) can be rewritten as

$$b + au + u \,\dot{\mathcal{D}}\, b + bu + u\,. \tag{2.18}$$

From (2.15), (2.16) and (2.18) we find

$$b + u \,\dot{\mathcal{D}}\, (a + u)(b + u)\,. \tag{2.19}$$

Putting $s = w + a$ and interchaning the role of a and b we similarly obtain

$$a + s \,\dot{\mathcal{D}}\, (b + s)(a + s)\,. \tag{2.20}$$

Using Lemma 2.2 we have from (2.14) that

$$a + w + a \,\dot{\mathcal{D}}\, a + w + b, \quad b + w + a \,\dot{\mathcal{D}}\, b + w + b$$

and thus

$$a + s \,\dot{\mathcal{D}}\, a + u\,, \; b + s \,\dot{\mathcal{D}}\, b + u\,. \tag{2.21}$$

Therefore (2.20) can be rewritten as

$$a + u \,\dot{\mathcal{D}}\, (b + u)(a + u) \tag{2.22}$$

and from (2.19) and (2.22) we infer that $a + u \,\dot{\mathcal{D}}\, b + u$. We found that

$$a + w + b \,\dot{\mathcal{D}}\, b + w + b\,. \tag{2.23}$$

The elements mentioned in (2.13) and (2.23) belong to the same $\overset{+}{\mathcal{D}}$-class, and since this $\overset{+}{\mathcal{D}}$-class forms a subsemiring for which $\dot{\mathcal{D}}$ is a congruence relation by Theorem 1.5 and Corollary 1.11, we have from (2.13) and (2.23) that

$$(a+w+b)+(w+a+w)\;\dot{\mathcal{D}}\;(b+w+b)+(w+b+w)$$

that is, $a+w\,\dot{\mathcal{D}}\,b+w$.

We proved that if $a(\dot{\mathcal{R}}\cap\overset{+}{\mathcal{R}})b$, then $a+w\;\dot{\mathcal{D}}\;b+w$. From this result and the left-right dual for the multiplication we infer in the usual way that if $a(\dot{\mathcal{D}}\cap\overset{+}{\mathcal{R}})\,b$, then $a+w\,\dot{\mathcal{D}}\,b+w$. □

We are now ready for our first conclusion.

Theorem 2.11. *$\dot{\mathcal{D}}$ is a congruence relation on every $S\in\mathbf{N}$.*

Proof. Let $a,b,w\in S$, with $S\in\mathbf{N}$, such that $a\;\dot{\mathcal{D}}\;b$. By Theorem 1.9 the elements $a,b,\ a+b+a,\ a+b$ and $b+a+b$ are all $\dot{\mathcal{D}}$-related. By Lemma 2.7 we have

$$a+w\,\dot{\mathcal{D}}\,a+b+a+w,\quad b+w\,\dot{\mathcal{D}}\,b+a+b+w\,. \tag{2.24}$$

By Lemma 2.1 we have

$$a+b+w\,\dot{\mathcal{D}}\,b+a+b+w\,. \tag{2.25}$$

By Lemma 2.10 we have

$$a+b+w\,\dot{\mathcal{D}}\,a+b+a+w\,. \tag{2.26}$$

From (2.24), (2.25) and (2.26) we thus find that $a+w\;\dot{\mathcal{D}}\;b+w$. The left-right dual for the addition follows similarly, thus $w+a\;\dot{\mathcal{D}}\;w+b$. Since evidently also $aw\,\dot{\mathcal{D}}\,bw\,\dot{\mathcal{D}}\,wb\,\dot{\mathcal{D}}\,wa$, we conclude that $\dot{\mathcal{D}}$ is a congruence. □

Lemma 2.12. *Let $S\in\mathbf{I}$ and $a,b\in S$.*

(i) *$a(\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}})\,b$ if and only if $\overset{+}{D}_a$ and $\overset{+}{D}_b$ are $\dot{\mathcal{D}}$-related in $S/\overset{+}{\mathcal{D}}$.*

(ii) *If $\dot{\mathcal{D}}$ is a congruence relation, then $a(\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}})\,b$ if and only if $\dot{D}_a$ and $\dot{D}_b$ are $\overset{+}{\mathcal{D}}$-related in $S/\dot{\mathcal{D}}$.*

Proof. We give a proof of (i) only. The proof of (ii) is similar. If $a\,\dot{\mathcal{D}}\,b$, then obviously $\overset{+}{D}_a$ and $\overset{+}{D}_b$ are $\dot{\mathcal{D}}$-related in $S/\overset{+}{\mathcal{D}}$. Since $\overset{+}{\mathcal{D}}\vee\dot{\mathcal{D}}$ is the transitive closure of $\overset{+}{\mathcal{D}}\cup\dot{\mathcal{D}}$ it then follows that if $a(\dot{\mathcal{D}}\vee\overset{+}{\mathcal{D}})b$, then again $\overset{+}{D}_a$ and $\overset{+}{D}_b$ are $\dot{\mathcal{D}}$-related in $S/\overset{+}{\mathcal{D}}$.

Assume that $\overset{+}{D}_a$ and $\overset{+}{D}_b$ are $\dot{\mathcal{D}}$-related in $S/\overset{+}{\mathcal{D}}$. Thus in $S/\overset{+}{\mathcal{D}}$ we have $\overset{+}{D}_a=\overset{+}{D}_a\,\overset{+}{D}_b\,\overset{+}{D}_a=\overset{+}{D}_{aba}$ and similarly $\overset{+}{D}_b=\overset{+}{D}_{bab}$. Therefore in S we have $a\,\overset{+}{\mathcal{D}}\,aba\,\dot{\mathcal{D}}\,bab\,\overset{+}{\mathcal{D}}\,b$. □

Lemma 2.13. *Let S be an idempotent semiring. Then $\dot{\mathcal{D}} \vee \overset{+}{\mathcal{D}}$ is a congruence on S if and only if the $\dot{\mathcal{D}}$-relation $\dot{\mathcal{D}}_{S/\overset{+}{\mathcal{D}}}$ on $S/\overset{+}{\mathcal{D}}$ is a congruence relation on $S/\overset{+}{\mathcal{D}}$. If this is the case, then $S/(\dot{\mathcal{D}} \vee \overset{+}{\mathcal{D}}) \cong (S/\overset{+}{\mathcal{D}})/\dot{\mathcal{D}}_{S/\overset{+}{\mathcal{D}}}$ is the greatest bisemilattice homomorphic image of S.*

Proof. The first statement follows immediately from Lemma 2.12, the second statement follows from Lemma 2.12 and Theorem 1.4. □

Lemma 2.14. *Let S be an idempotent semiring on which $\dot{\mathcal{D}}$ is a congruence. Then $\dot{\mathcal{D}} \vee \overset{+}{\mathcal{D}}$ is the least* **Bi***-congruence on S, the $\overset{+}{\mathcal{D}}$-relation $\overset{+}{\mathcal{D}}_{S/\dot{\mathcal{D}}}$ on $S/\dot{\mathcal{D}}$ is the least* **Bi***-congruence on $S/\dot{\mathcal{D}}$ and $S/(\dot{\mathcal{D}} \vee \overset{+}{\mathcal{D}}) \cong (S/\dot{\mathcal{D}})/\overset{+}{\mathcal{D}}_{S/\dot{\mathcal{D}}}$ is the greatest bisemilattice homomorphic image of S.*

Proof. Immediate from Lemma 2.12 and Theorem 1.4. □

Lemma 2.15. *Let S be an idempotent semiring and ρ a congruence on S such that $\rho \subseteq \overset{+}{\mathcal{D}}$ or $\rho \subseteq \dot{\mathcal{D}}$. Then S contains a two-element monobisemilattice if and only if S/ρ does.*

Proof. If S contains a two-element monobisemilattice $T = \{a, b\}$, then a and b are neither $\dot{\mathcal{D}}$- nor $\overset{+}{\mathcal{D}}$-related and so $\{a\rho, b\rho\}$ forms a two-element monobisemilattice. Conversely, if $a, b \in S$ are such that $\{a\rho, b\rho\}$ forms a two-element monobisemilattice where $a\rho \leq_{\cdot} b\rho$ and $a\rho \;{}_{+}\!\geq b\rho$ in S/ρ, then $\{b,\ b + bab + b\}$ forms a monobisemilattice in S and $b \neq b + bab + b$ since $b\rho \neq a\rho = (b + bab + b)\,\rho$. □

The following complements Theorem 1.1.

Theorem 2.16. *For an idempotent semiring S the following are equivalent:*

(i) *$S \in \mathbf{N}$,*

(ii) *$\dot{\mathcal{D}} \vee \overset{+}{\mathcal{D}}$ is the least lattice congruence on S,*

(iii) *$\dot{\mathcal{D}}_{S/\overset{+}{\mathcal{D}}}$ is the least lattice congruence on $S/\overset{+}{\mathcal{D}}$,*

(iv) *$\dot{\mathcal{D}}$ is a congruence on S and $\overset{+}{\mathcal{D}}_{S/\dot{\mathcal{D}}}$ is the least lattice congruence on $S/\dot{\mathcal{D}}$,*

(v) *$\leq_{+} \cap \;{}_{\cdot}\!\geq$ is the equality on S.*

Proof. That (i) and (v) are equivalent follows from Corollary 1.3. The equivalence of (ii) and (iii) follows from Lemma 2.13 and Theorem 1.4.

Assume that $S \in \mathbf{N}$. Then $\dot{\mathcal{D}}$ is a congruence relation on S by Theorem 2.11 and so $\overset{+}{\mathcal{D}}_{S/\dot{\mathcal{D}}}$ is the least bisemilattice congruence on $S/\dot{\mathcal{D}}$ by Lemma 2.14. By Theorem 1.1, $S/\dot{\mathcal{D}}$ and $(S/\dot{\mathcal{D}})/\overset{+}{\mathcal{D}}_{S/\dot{\mathcal{D}}}$ do not contain a two-element monobisemilattice. Therefore $\overset{+}{\mathcal{D}}_{S/\dot{\mathcal{D}}}$ is the least lattice congruence. We proved that (i) implies (iv).

Assume that (iv) holds. Then (ii) holds in view of Lemma 2.14. If (iii) holds, then $S/\overset{+}{\mathcal{D}}$ does not contain a two-element monobisemilattice in view of Lemma

2.15 and therefore S does not contain a two-element monobisemilattice, again by Lemma 2.15. Thus $S \in \mathbf{N}$ by Theorem 1.1. Thus (iii) implies (i). $\square$

We now look at a special case.

Theorem 2.17. *For an idempotent semiring S the following are equivalent:*
(i) $\dot{\mathcal{D}}$ *is the least lattice congruence on* S,
(ii) $S \in \mathbf{N}$ *and* $\overset{+}{\mathcal{D}} \subseteq \dot{\mathcal{D}}$ *for* S,
(iii) $\leq_{\cdot} \subseteq \leq_{+}$ *for* S,
(iv) S *satisfies the identity*

$$x \approx xyx + x + xyx\,. \tag{2.27}$$

Proof. The equivalence of (iii) and (iv) was proved in [4]. If (i) is satisfied, then $S \in \mathbf{N}$ by Theorem 1.1 and Lemma 2.15, and $\overset{+}{\mathcal{D}} \subseteq \dot{\mathcal{D}}$ for S since $\overset{+}{\mathcal{D}}$ is the least semilattice congruence on the additive reduct. Therefore (i) implies (ii). That (ii) implies (i) follows from Theorem 2.16. We proved that (i) and (ii) are equivalent.

If (i) holds, then the absorption law in $S/\dot{\mathcal{D}}$ yields that for every $a, b \in S$, we have

$$\dot{D}_a = \dot{D}_a + \dot{D}_a\,\dot{D}_b$$

in $S/\dot{\mathcal{D}}$. Therefore also

$$\dot{D}_a = \dot{D}_a\,\dot{D}_b\,\dot{D}_a + \dot{D}_a + \dot{D}_a\,\dot{D}_b\,\dot{D}_a = \dot{D}_{aba+a+aba}$$

in $S/\dot{\mathcal{D}}$ and so $a\,\dot{\mathcal{D}}\,(aba + a + aba)$ in S. Hence for all $a, b \in S$

$$a = a(aba + a + aba)a = aba + a + aba$$

and so S satisfies the identity (2.27). We proved that (i) implies (iv).

Assume that (iv) holds. Then $S \in \mathbf{N}$ by Theorem 1.1. Let $a\,\overset{+}{\mathcal{D}}\,b$ in S, thus $b = b + a + b$ and so $aba = aba + a + aba = a$, and similarly $bab = b$, whence $a\,\dot{\mathcal{D}}\,b$. Therefore $\overset{+}{\mathcal{D}} \subseteq \dot{\mathcal{D}}$. We proved that (iv) implies (ii). $\square$

It is easy to see that the idempotent semirings for which $\overset{+}{\mathcal{D}} \subseteq \dot{\mathcal{D}}$ form a subvariety of $\mathbf{I}$ determined by the identity

$$(x + y)(y + x)(x + y) \approx x + y\,. \tag{2.28}$$

Therefore the variety consisting of the idempotent semirings satisfying the equivalent conditions of Theorem 2.17 is also determined by the identities (1.7) and (2.28). In view of Theorem 1.1 this variety is also determined by

$$x \approx x + xyx \approx xyx + x\,. \tag{2.29}$$

In view of Example 1.6, the idempotent semirings for which $\overset{+}{\mathcal{D}} \subseteq \dot{\mathcal{D}}$ do not necessarily have the property that $\dot{\mathcal{D}}$ is a congruence. Therefore the idempotent semirings that satisfy the equivalent conditions of Theorem 2.17 form a proper subvariety of the subvariety of $\mathbf{I}$ determined by (2.28).

The dual of Theorem 2.17 is also true:

Theorem 2.18. [4] *For an idempotent semiring S the following are equivalent:*

(i) $\overset{+}{\mathcal{D}}$ *is the least lattice congruence on S,*

(ii) $S \in \mathbf{N}$ *and* $\dot{\mathcal{D}} \subseteq \overset{+}{\mathcal{D}}$ *for S,*

(iii) $\leq_{+} \subseteq \leq_{\cdot}$ *for S,*

(iv) *S satisfies the identity*

$$x = (x + y + x)x(x + y + x)\,. \tag{2.30}$$

We remark at once that while our proof of Theorem 2.17 could be dualized to prove Theorem 2.18, the dual of the proof in [4] for the results of Theorem 2.18 is not valid.

For subvarieties $\mathbf{V}$ and $\mathbf{W}$ of $\mathbf{I}$, the *Mal'cev product* $\mathbf{V} \circ \mathbf{W}$ of $\mathbf{V}$ and $\mathbf{W}$ (within $\mathbf{I}$) is the class of all idempotent semirings S on which there exists a congruence ρ such that $S/\rho \in \mathbf{W}$ and such that the ρ-classes belong to $\mathbf{V}$. We denote the variety consisting of the idempotent semirings whose additive [multiplicative] reduct is a rectangular band by $\overset{+}{\mathbf{R}}\,[\dot{\mathbf{R}}]$. Part (iii) of the following was proved in [4].

Theorem 2.19. (i) $\mathbf{N} = \overset{+}{\mathbf{R}} \circ \left(\dot{\mathbf{R}} \circ \mathbf{D}\right) = \dot{\mathbf{R}} \circ \left(\overset{+}{\mathbf{R}} \circ \mathbf{D}\right)$,

(ii) $\dot{\mathbf{R}} \circ \mathbf{D}$ *is the variety consisting of the idempotent semirings satisfying the equivalent conditions of Theorem 2.17.*

(iii) $\overset{+}{\mathbf{R}} \circ \mathbf{D}$ *is the variety consisting of the idempotent semirings satisfying the equivalent conditions of Theorem 2.18.*

Proof. (i) Let $S \in \mathbf{N}$. Then by Theorem 2.16, $S/\overset{+}{\mathcal{D}} \in \dot{\mathbf{R}} \circ \mathbf{D}$ and $S/\dot{\mathcal{D}} \in \overset{+}{\mathbf{R}} \circ \mathbf{D}$, whence $\mathbf{N} \subseteq \overset{+}{\mathbf{R}} \circ (\dot{\mathbf{R}} \circ \mathbf{D})$ and $\mathbf{N} \subseteq \dot{\mathbf{R}} \circ \left(\overset{+}{\mathbf{R}} \circ \mathbf{D}\right)$. If $S \in \overset{+}{\mathbf{R}} \circ (\dot{\mathbf{R}} \circ \mathbf{D})$ then there exist congruences ρ and σ on S and S/ρ respectively such that $(S/\rho)/\sigma \in \mathbf{D}$, $\sigma \subseteq \dot{\mathcal{D}}_{S/\rho}$ and $\rho \subseteq \overset{+}{\mathcal{D}}$. By Lemma 2.15 neither S/ρ nor S can contain a two-element monobisemilattice and therefore $S \in \mathbf{N}$ by Theorem 1.1. Thus $\mathbf{N} = \overset{+}{\mathbf{R}} \circ \left(\dot{\mathbf{R}} \circ \mathbf{D}\right)$ and similarly $\mathbf{N} = \dot{\mathbf{R}} \circ \left(\overset{+}{\mathbf{R}} \circ \mathbf{D}\right)$.

(ii), (iii). Immediate from Theorems 2.17 and 2.18 by using 2.15 and 2.16. $\square$

The lattices $L(\mathbf{N} \cap \mathbf{ID})$, $L\left(\left(\overset{+}{\mathbf{R}} \circ \mathbf{D}\right) \cap \mathbf{ID}\right)$ and $L\left(\left(\dot{\mathbf{R}} \circ \mathbf{D}\right) \cap \mathbf{ID}\right)$ of subvarieties of $\mathbf{N} \cap \mathbf{ID}$, $\left(\overset{+}{\mathbf{R}} \circ \mathbf{D}\right) \cap \mathbf{ID}$ and $\left(\dot{\mathbf{R}} \circ \mathbf{D}\right) \cap \mathbf{ID}$ respectively were described in [3]. $L(\mathbf{N} \cap \mathbf{ID})$ is isomorphic to the direct product of a two-element lattice and a lattice which is itself a subdirect product of four copies of the lattice of all band varieties. One verifies easily from the results of [3] that both $L\left(\left(\overset{+}{\mathbf{R}} \circ \mathbf{D}\right) \cap \mathbf{ID}\right)$ and $L\left(\left(\dot{\mathbf{R}} \circ \mathbf{ID}\right) \cap \mathbf{ID}\right)$ are isomorphic to the direct product of a two-element lattice and a lattice which is itself a direct product of two copies of the lattice of all band varieties. We remark that the semiring of Example 1.12 belongs to $\mathbf{N}$, but not to $\mathbf{N} \cap \mathbf{ID}$. In fact, this semiring belongs to $\dot{\mathbf{R}} \circ \mathbf{D}$ but not to $(\dot{\mathbf{R}} \circ \mathbf{D}) \cap \mathbf{ID}$.

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