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SMOOTH BUNDLES OF GENERALIZED HALF-DENSITIES

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Abstract. Smooth bundles, whose fibres are distribution spaces, are introduced according to the notion of smoothness due to Frölicher. Some fundamental notions of differential geometry, such as tangent and jet spaces, Frölicher-Nijenhuis bracket, connections and curvature, are suitably generalized. It is also shown that a classical connection on a finite-dimensional bundle naturally determines a connection on an associated distributional bundle.

1. Introduction

Let $\mathcal{M}$ be any set and $\mathfrak{C}_{\mathcal{M}}$ a set of curves $\mathbb{R} \to \mathcal{M}$. Then, $\mathfrak{C}_{\mathcal{M}}$ determines a set $\mathfrak{F}\mathfrak{C}_{\mathcal{M}}$ of maps $\mathcal{M} \to \mathbb{R}$ according to

$$f \in \mathfrak{F}\mathfrak{C}_{\mathcal{M}} \iff f \circ c \in \mathcal{C}^{\infty}(\mathbb{R})\ \forall\, c \in \mathfrak{C}_{\mathcal{M}}\,.$$

Conversely, a set $\mathfrak{F}_{\mathcal{M}}$ of functions $\mathcal{M} \to \mathbb{R}$ determines a set $\mathfrak{C}\mathfrak{F}_{\mathcal{M}}$ of curves in $\mathcal{M}$ according to

$$c \in \mathfrak{C}\mathfrak{F}_{\mathcal{M}} \iff f \circ c \in \mathcal{C}^{\infty}(\mathbb{R})\ \forall\, f \in \mathfrak{F}_{\mathcal{M}}\,.$$

Following Frölicher [Fr82], a *smooth structure* on $\mathcal{M}$ is defined to be a couple $(\mathfrak{C}_{\mathcal{M}}, \mathfrak{F}_{\mathcal{M}})$ such that $\mathfrak{C}_{\mathcal{M}}$ and $\mathfrak{F}_{\mathcal{M}}$ determine each other, namely

$$\mathfrak{F}\mathfrak{C}_{\mathcal{M}} = \mathfrak{F}_{\mathcal{M}}\,, \quad \mathfrak{C}\mathfrak{F}_{\mathcal{M}} = \mathfrak{C}_{\mathcal{M}}\,.$$

For clarity we call this an *F-smooth structure*. By abuse of language we may also call $\mathcal{M}$ an F-smooth space. Note that any set $\mathfrak{C}_0$ of curves in $\mathcal{M}$, or any set $\mathfrak{F}_0$ of functions on $\mathcal{M}$, *generate* an F-smooth structure by $\mathfrak{F}_{\mathcal{M}} := \mathfrak{F}\mathfrak{C}_0$ or $\mathfrak{C}_{\mathcal{M}} := \mathfrak{C}\mathfrak{F}_0$.

If $(\mathcal{N}, \mathfrak{C}_{\mathcal{N}}, \mathfrak{F}_{\mathcal{N}})$ is another F-smooth structure, then a map $\Phi : \mathcal{M} \to \mathcal{N}$ is called F-smooth if $\Phi \circ c \in \mathfrak{C}_{\mathcal{N}}$ for all $c \in \mathfrak{C}_{\mathcal{M}}$, or equivalently if $f \circ \Phi \in \mathfrak{F}_{\mathcal{M}}$ for all $f \in \mathfrak{F}_{\mathcal{N}}$. F-smoothness behaves naturally with regard to cartesian products and inclusion.

The notion of F-smoothness provides a setting for a general approach to calculus in infinite dimensional spaces [FK88, KM97]. For Banach spaces, and manifolds modelled on them, one recovers the usual smooth structure. It has also been observed that there are important situations in which a simplified version of Frölicher's approach is sufficient to develop several basic ideas of differential geometry: one has only to consider a suitable reduced set $\mathfrak{C}_0$ of curves in $\mathcal{M}$, fulfilling a few consistency axioms. In this way, notions such as tangent spaces, jet spaces, connections and curvature, have been introduced and studied for a large class of functional

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bundles [JM, MK98, CK]. Everything can be formulated in terms of smooth classical maps, without getting involved in the topologies of functional spaces and other intricated questions (such as the characterization of the whole set of F-smooth maps).

In this paper I'll adhere to that same basic philosophy in studying distributional bundles, and in particular bundles of generalized half-densities (in view of applications to covariant QFT). But now the criterium for F-smoothness has to be introduced in a somewhat different way: a space $\mathcal{U}$ of 'test' maps is taken to be the set $\mathfrak{F}_0$ generating the F-smooth structure on its topological dual $\mathcal{U}'$. We shall be only slightly involved in topological questions, however.

Unless otherwise explicitely said, by 'manifold' or 'classical manifold' I shall always mean a smooth Hausdorff paracompact manifold of finite dimension. Classical manifolds will be indicated by boldface latin capital letters: $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{M}$ and so on. Infinite dimensional spaces will be indicated by calligraphic capital letters like $\mathcal{X}$, $\mathcal{Y}$ and so on.

2. Generalized half-densities

Throughout this paper, by $\boldsymbol{Y}$ we shall denote an oriented real classical manifold of dimension n. We set $\mathbb{V}\boldsymbol{Y} := (\wedge^n \mathrm{T}\boldsymbol{Y})^+$ (or simply $\mathbb{V}$ if no confusion arises), so that $\mathbb{V}^{-1} := \mathbb{V}^*$ stands for the half-vector bundle of positive densities (i.e. volume forms). Then $\mathbb{V}^{-1/2} := (\mathbb{V}^{1/2})^*$ is the half-vector bundle of the so-called *half-densities*.

Let $\boldsymbol{V} \to \boldsymbol{Y}$ be a classical vector bundle; we denote by $\mathcal{U}(\boldsymbol{Y}, \boldsymbol{V})$ the vector space of all global smooth sections $u : \boldsymbol{Y} \to \boldsymbol{V}$ which have compact support. A topology on this space can be introduced by the standard procedure [Sc66]; $\mathcal{U}(\boldsymbol{Y}, \boldsymbol{V})$ is a complete countably normed space. Its topological dual will be denoted by $\mathcal{U}'(\boldsymbol{Y}, \boldsymbol{V})$; in particular, a sufficiently regular (for instance, smooth) section $\theta : \boldsymbol{Y} \to \boldsymbol{V}^* \otimes_{\boldsymbol{Y}} \mathbb{V}^{-1}\boldsymbol{Y}$ can be seen as an element of $\mathcal{U}'(\boldsymbol{Y}, \boldsymbol{V})$ by the rule

$$\langle \theta, u \rangle := \int_{\boldsymbol{Y}} \langle \theta(y), u(y) \rangle \,.$$

In general if $\theta \in \mathcal{U}'(\boldsymbol{Y}, \boldsymbol{V})$ we shall also write

$$\theta : \boldsymbol{Y} \rightsquigarrow \boldsymbol{V}^* \underset{\boldsymbol{Y}}{\otimes} \mathbb{V}^{-1}\boldsymbol{Y}$$

and call it a generalized section. We stress that generalized sections can be naturally restricted to any open subset of $\boldsymbol{Y}$, and that a gluing property holds: if $\{\boldsymbol{Y}_i\}$ is an open covering of $\boldsymbol{Y}$ and $\{\theta_i \in \mathcal{U}'(\boldsymbol{Y}_i\,, \boldsymbol{V})\}$ is a family of generalized sections, such that θ_i and θ_j coincide on $\boldsymbol{Y}_i \cap \boldsymbol{Y}_j$ whenever this is non-empty, then there is a unique $\theta \in \mathcal{Y}$ whose restriction to $\boldsymbol{Y}_i$ coincides with $\theta_i \;\forall\, i$.

We shall be particularly involved with the space of *generalized complex half-densities*

$$\mathcal{Y} := \mathcal{U}'(\boldsymbol{Y}, \mathbb{C} \otimes \mathbb{V}^{-1/2}) := \mathcal{Y}_{\circ}{}' \,.$$

Namely $\mathcal{Y}_{\circ}$, the space of smooth complex half-densities with compact support, is a subspace of $\mathcal{Y}$; it can be shown that $\mathcal{Y}_{\circ}$ is dense in $\mathcal{Y}$. If $\boldsymbol{V}$ is a trivial (not just trivializable) complex bundle, then $\mathcal{U}'(\boldsymbol{Y}, \boldsymbol{V} \otimes \mathbb{V}^{-1/2}) \equiv \boldsymbol{V}^* \otimes \mathcal{Y}$.

Consider now an orientation-preserving diffeomorphism $\varphi : \boldsymbol{Y} \to \boldsymbol{Z}$ between two oriented manifolds; it determines a linear isomorphism $\varphi_* := (\varphi^{-1})^* : \wedge \mathrm{T}^*\boldsymbol{Y} \to \wedge \mathrm{T}^*\boldsymbol{Z}$ (over φ) of the corresponding exterior algebras, whose square root, denoted again with the same symbol, is an isomorphism of the corresponding half-density bundles, given by

$$\varphi_* : \mathbb{V}^{-1/2}\boldsymbol{Y} \to \mathbb{V}^{-1/2}\boldsymbol{Z} : \lambda \mapsto \sqrt{\varphi_*(\lambda^2)}\,.$$

The maps φ_* , on turn, determine isomorphisms of the corresponding sheaves of exterior algebras and half-densities, denoted by the same symbol. In particular, $\varphi_* : \mathcal{Y}_\circ \to \mathcal{Z}_\circ$. Now φ_* extends to generalized half-densities as

$$\langle \varphi_*\lambda, \varphi_* u\rangle = \langle \lambda, u\rangle\,, \quad \forall\, \lambda \in \mathcal{Y},\ u \in \mathcal{Y}_\circ\,.$$

It is easy to see that $\varphi_* : \mathcal{Y} \to \mathcal{Z}$ is a continuous linear isomorphism, and that the correspondence $\varphi \mapsto \varphi_*$ behaves naturally with regard to compositions.

Let $v : \boldsymbol{Y} \to \mathrm{T}\boldsymbol{Y}$ be a smooth vector field. If $\lambda \in \mathcal{Y}$ is a nowhere vanishing *smooth* half-density, then the Lie derivative $v.\lambda$ is naturally defined as

$$v.\lambda := \frac{1}{2\lambda}\, v.(\lambda^2) = \frac{1}{2\lambda}\, \mathrm{d}(v \,|\, \lambda^2)\,,$$

where $(v \,|\, \lambda^2)$ denotes standard contraction. Moreover, this naturally extends to all smooth half-densities (consider local expressions) as $v.(f\lambda) := (v.f)\lambda + f(v.\lambda)$, where $f : \boldsymbol{Y} \to \mathbb{C}$ is any smooth function. If moreover $u \in \mathcal{Y}_\circ$ then $v.(\lambda \otimes u) = (v.\lambda) \otimes u + \lambda \otimes (v.u)$, so that

$$\int_{\boldsymbol{Y}} (v.\lambda) \otimes u = \int_{\boldsymbol{Y}} \mathrm{d}\big(v \mid (\lambda \otimes u)\big) - \int_{\boldsymbol{Y}} \lambda \otimes (v.u)\,.$$

Since the first term in the right hand-side vanishes (because u has compact support) we obtain

$$\langle v.\lambda, u\rangle = -\langle \lambda, v.u\rangle\,,$$

which can be taken as the *definition* of $v.\lambda$ whenever λ is a generalized half-density. Then the map $\lambda \mapsto v.\lambda$ turns out to be a continuous linear operator in $\mathcal{Y}$.

Let now $\check{\boldsymbol{Y}} \subset \boldsymbol{Y}$ be an open subset, and $\mathsf{y} \equiv (\mathsf{y}^i) : \check{\boldsymbol{Y}} \to \boldsymbol{A} \subset \mathbb{R}^n$ a coordinate chart. Denote by $\mathcal{A}$ and $\mathcal{R}^n$ the spaces of generalized complex half-densities on $\boldsymbol{A}$ and $\mathbb{R}^n$, respectively; we have a canonical inclusion $\mathcal{A} \subset \mathcal{R}^n$ (extend an element in $\mathcal{A}$ by letting it vanish outside $\boldsymbol{A}$) and we obtain the induced 'coordinate' chart $\mathsf{y}_* : \check{\mathcal{Y}} \to \mathcal{A} \subset \mathcal{R}^n$. In $\mathcal{A}$ we have a canonical half-density, the square root of the canonical positive volume form of $\mathbb{R}^n$, so that $\mathcal{A}$ can be identified with the space of generalized maps $\boldsymbol{A} \rightsquigarrow \mathbb{C}$.

For $\lambda \in \mathcal{Y}$ we set

$$\lambda_{\mathsf{y}} := \mathsf{y}_*\lambda \in \mathcal{A} \subset \mathcal{R}^n\,,$$

which is analogous to the set of components of a vector in a finite dimensional space.

Consider again an orientation-preserving diffeomorphism $\varphi : \boldsymbol{Y} \to \boldsymbol{Z}$ and a local chart $\mathsf{z} : \varphi(\check{\boldsymbol{Y}}) \to \boldsymbol{B} \subset \mathbb{R}^n$, and set

$$\varphi_{\mathsf{zy}} := \mathsf{z} \circ \varphi \circ \mathsf{y}^{-1} : \boldsymbol{A} \to \boldsymbol{B}\,.$$

Then $\mathrm{D}\varphi_{\mathsf{zy}} : \boldsymbol{A} \to \mathbb{R}^{n^2}$. Moreover we indicate by $|\varphi_{\mathsf{zy}}| := \det(\mathrm{D}\varphi_{\mathsf{zy}}) : \boldsymbol{A} \to \mathbb{R}^+$ the Jacobian determinant of φ_{zy}. We write the 'coordinate' expression of φ_* as

$$(\varphi_*\lambda)_{\mathsf{z}} = (\lambda_{\mathsf{y}} \circ \varphi_{\mathsf{yz}}^{-1}) \sqrt{|\varphi_{\mathsf{yz}}^{-1}|}\,.$$

In particular, if φ is just the identity map of $\check{\boldsymbol{Y}}$, we obtain the coordinate transformation formula

$$\lambda_{\mathsf{z}} = \lambda_{\mathsf{y}} \sqrt{|\kappa_{\mathsf{yz}}|}\,, \quad \kappa_{\mathsf{yz}} := (1\!\!1_{\boldsymbol{Y}})_{\mathsf{yz}}\,.$$

Consider now the vector field $\partial \mathsf{y}_i$ on $\check{\boldsymbol{Y}}$ induced by the coordinates, $1 \leq i \leq n$. We write

$$\partial_i \lambda := \partial \mathsf{y}_i . \lambda \in \check{\mathcal{Y}} \quad \Rightarrow \quad \partial_i \lambda_{\mathsf{y}} = (\partial_i \lambda)_{\mathsf{y}} : \boldsymbol{A} \rightsquigarrow \mathbb{C}.$$

If $v : \boldsymbol{Y} \to \mathrm{T}\boldsymbol{Y}$ is a vector field then we obtain

$$(v.\lambda)_{\mathsf{y}} = v^i \partial_i \lambda_{\mathsf{y}} + \tfrac{1}{2}\,(\partial_i v^i)\lambda_{\mathsf{y}}\,.$$

Let $\mathcal{L}^2\boldsymbol{Y}$ be the space of all complex half-densities $\lambda : \boldsymbol{Y} \to \mathbb{C} \otimes \mathbb{V}^{-1/2}$ such that

$$\|\lambda\|^2 := \int_{\boldsymbol{Y}} |\lambda|^2 < \infty\,,$$

and $\tilde{0}\boldsymbol{Y}$ the subspace of all almost-everywhere vanishing half-densities. Then $\mathcal{H} \equiv \mathcal{H}\boldsymbol{Y} := \mathcal{L}^2\boldsymbol{Y}/\tilde{0}\boldsymbol{Y}$ turns out to be a Hilbert space with the Hermitian product given by

$$\langle \lambda, \mu \rangle := \int_{\boldsymbol{Y}} \bar{\lambda}\,\mu\,.$$

We have

$$\mathcal{Y}_\circ \subset \mathcal{H} \cong \mathcal{H}' \subset \mathcal{Y}\,,$$

namely the triple $(\mathcal{Y}_\circ, \mathcal{H}, \mathcal{Y})$ constitutes a so-called *rigged Hilbert space* [BLT75]. Note that $\mathcal{Y}_\circ$ is incomplete in the $\mathcal{L}^2$ toplogy and dense in $\mathcal{H}$.

If the fibres of the vector bundle $\boldsymbol{V} \to \boldsymbol{Y}$ are smoothly endowed with a Hermitian structure, then the above settings can be easily extended in order to describe the rigged Hilbert space of (generalized) sections $\boldsymbol{Y} \rightsquigarrow \boldsymbol{V} \otimes_{\boldsymbol{Y}} \mathbb{V}^{-1/2}$.

3. F-smoothness

We shall consider on $\mathcal{Y}$ the F-smooth structure generated by $\mathcal{Y}_\circ$, seen as a set of (linear) maps $\mathcal{Y} \to \mathbb{C}$. Namely, let $\mathbb{I} \subset \mathbb{R}$ be an open interval; then a curve $\alpha : \mathbb{I} \to \mathcal{Y}$ will be called F-smooth if the map

$$\langle \alpha, u \rangle : \mathbb{I} \to \mathbb{C} : t \mapsto \langle \alpha(t), u \rangle$$

is smooth for all $u \in \mathcal{Y}_\circ$.

Let $\mathfrak{C}_{\mathcal{Y}}$ be the set of all F-smooth curves in $\mathcal{Y}$; take any $p \in \mathbb{N} \cup \{0\}$ and consider the following binary relation on $\mathbb{R} \times \mathfrak{C}_{\mathcal{Y}}$:

$$(t, \alpha) \overset{p}{\sim} (s, \beta) \iff \mathrm{D}^k \langle \alpha, u \rangle(t) = \mathrm{D}^k \langle \beta, u \rangle(s) \qquad \forall u \in \mathcal{Y}_\circ,\ k = 0, \ldots, p\,.$$

Then clearly $\overset{p}{\sim}$ is an equivalence relation; the quotient

$$\mathrm{T}^p\mathcal{Y} := \mathfrak{C}_{\mathcal{Y}} / \overset{p}{\sim}$$

will be called the *tangent space of order* p of $\mathcal{Y}$. The equivalence class of $(t,\alpha)\in\mathfrak{C}_{\mathcal{Y}}$ will be denoted by $\partial^p\alpha(t)$. Obviuosly, $\mathrm{T}^p\mathcal{Y}$ is a fibred set over $\mathcal{Y}$; the fibre over some $\lambda\in\mathcal{Y}$ will be denoted by $\mathrm{T}^p_\lambda\mathcal{Y}$. In particular $\mathrm{T}^0\mathcal{Y}=\mathcal{Y}$.

The set $\mathrm{T}\mathcal{Y}:=\mathrm{T}^1\mathcal{Y}$ is called simply the tangent space of $\mathcal{Y}$, and $\partial\alpha(t):=\partial^1\alpha(t)$ is called the *tangent vector of* α *at* $\alpha(t)$. Any element in $\mathrm{T}\mathcal{Y}$ can be represented as $\partial\alpha(0)$, for a suitable curve α defined on a neighbourhood $\mathbb{I}$ of 0.

Proposition 3.1. *For each $\lambda\in\mathcal{Y}$, $\mathrm{T}_\lambda\mathcal{Y}$ turns out to be a vector space by setting*

$$r\,\partial\alpha(0):=\partial[\alpha(rt)]_{t=0}\,,\quad r\in\mathbb{R}\,,$$
$$\partial\alpha(0)+\partial\beta(0):=\partial(\alpha+\beta-\lambda)(0)\,,\quad \alpha(0)=\beta(0)=\lambda\,.$$

Moreover the map

$$\mathcal{Y}\times\mathcal{Y}\to\mathrm{T}\mathcal{Y}:(\lambda,\mu)\mapsto\partial[\lambda+t\mu]_{t=0}$$

is an isomorphism.

Proof. It is clear that the operations of product by numbers and of sum ‘pass to the quotient’, so that they define a vector space structure on $\mathrm{T}_\lambda\mathcal{Y}$. Next we observe (see [Sc66], Ch.III, Th.XIII) that there is a unique $\alpha'\in\mathcal{Y}$ such that, for all $u\in\mathcal{Y}_\circ$,

$$\langle\alpha',u\rangle=\lim_{t\to0}\langle\tfrac{1}{t}\big(\alpha(t)-\alpha(0)\big),u\rangle=\mathrm{D}\langle\alpha,u\rangle(0)$$

so that there is a natural linear injection $\mathrm{T}\mathcal{Y}\hookrightarrow\mathcal{Y}\times\mathcal{Y}$. By considering the equivalence classes of all affine curves, one sees that this map is also surjective.

□

Similarly one sees that there is a natural isomorphism

$$\mathrm{T}^p\mathcal{Y}\cong\mathcal{Y}^{p+1}\,.$$

The notion of F-smoothnes in $\mathrm{T}^p\mathcal{Y}$ is now reduced to that in $\mathcal{Y}^{p+1}$.

Note that if a F-smooth curve is valued in $\mathcal{Y}_\circ$, then its tangent vector is not valued in $\mathcal{Y}_\circ$ in general.

Remark. Suppose that $\alpha:\mathbb{I}\to\mathcal{Y}$ can be represented as a map $\alpha:\mathbb{I}\times\boldsymbol{Y}\to\mathbb{C}\otimes\mathbb{V}^{-1/2}$, so that for each $t\in\mathbb{I}$ and $u\in\mathcal{Y}_\circ$ we have

$$\langle\alpha(t),u\rangle=\int_{\boldsymbol{Y}}\bar\alpha(t,y)\,u(y)\,.$$

If $\alpha(t,y)$ is well-behaved enough that one can differentiate under the integral sign, then α is F-smooth, and $\partial\alpha$ is represented by the map

$$\frac{\partial\alpha}{\partial t}:\mathbb{I}\times\boldsymbol{Y}\to\mathbb{C}\otimes\mathbb{V}^{-1/2}.$$

4. Tangent prolongations

Let $\boldsymbol{M}$ be a classical manifold. A map $\phi:\boldsymbol{M}\to\mathcal{Y}$ is called F-smooth if $\phi\circ c:\mathbb{I}\to\mathcal{Y}$ is F-smooth for every smooth curve $c:\mathbb{I}\to\boldsymbol{M}$; recalling the basic result by Boman [Bo67], we see that the F-smoothness of ϕ is equivalent to the smoothness of all of the maps $\phi_u:\boldsymbol{M}\to\mathbb{C}:x\mapsto\langle\phi(x),u\rangle$, $u\in\mathcal{Y}_\circ$.

Proposition 4.1. *If $\phi : \boldsymbol{M} \to \mathcal{Y}$ is F-smooth, then there is a unique map*

$$\mathrm{T}\phi \cong (\phi \circ \pi_{\boldsymbol{M}}, \mathrm{d}\phi) : \mathrm{T}\boldsymbol{M} \to \mathcal{Y} \times \mathcal{Y} = \mathrm{T}\mathcal{Y},$$

called the tangent prolongation of ϕ, *such that* $\mathrm{T}\phi \circ \partial c = \partial(\phi \circ c)$ *for every local smooth curve* $c : \mathbb{R} \to \boldsymbol{M}$*. Moreover* $\mathrm{T}\phi$ *is linear over* ϕ *and F-smooth.*

Proof. For each $u \in \mathcal{Y}_\circ$ the map $\phi_u : \boldsymbol{M} \to \mathbb{C}$ is smooth, hence $\mathrm{d}(\phi_u) : \mathrm{T}\boldsymbol{M} \to \mathbb{C}$ is smooth and linear. Since $\mathrm{d}(\phi_u)(\partial c(0)) = \mathrm{D}(\phi_u \circ c)(0) = \mathrm{D}\langle \phi \circ c, u\rangle(0) = \langle \partial(\phi \circ c)(0), u\rangle \equiv \langle \mathrm{d}\phi(\partial c(0)), u\rangle$, setting $(\mathrm{d}\phi)_u := \mathrm{d}(\phi_u)$ we obtain the stated map. □

Let $\mathsf{x} = (\mathsf{x}^a) : \boldsymbol{X} \to \mathbb{R}^m$ be a local coordinate chart, $\boldsymbol{X} \subset \boldsymbol{M}$, and $\mathsf{x}_a : \mathbb{R} \to \boldsymbol{M}$ any coordinate curve. We define the partial derivative $\partial_a\phi$ at the point $\mathsf{x}_a(0)$ to be $\partial(\phi \circ \mathsf{x}_a)(0)$; we obtain a map $\partial_a\phi : \boldsymbol{X} \to \mathrm{T}\mathcal{Y}$. Locally we have

$$\mathrm{T}\phi = \partial_a\phi\, \mathrm{d}\mathsf{x}^a, \text{ i.e. } \langle \mathrm{d}\phi, u\rangle = \langle \partial_a\phi, u\rangle\, \mathrm{d}\mathsf{x}^a, \quad u \in \mathcal{Y}_\circ.$$

Next we are concerned with the tangent prolongations of F-smooth functions on $\mathcal{Y}$, which is less immediate. If $\boldsymbol{N}$ is a classical manifold, then a map $f : \mathcal{Y} \to \boldsymbol{N}$ is called F-smooth if $f \circ \alpha : \mathbb{I} \to \mathbb{R}$ is smooth for all F-smooth curves $\alpha : \mathbb{I} \to \mathcal{Y}$ (we remark that the F-smoothness of f does not allow any statement about its continuity with respect to the standard distribution space topology).

Lemma 4.1. *If $f : \mathcal{Y} \to \mathbb{R}$, $\alpha : \mathbb{I} \to \mathcal{Y}$ are F-smooth, and $\alpha(t) = o(t)$, then $\mathrm{D}(f \circ \alpha)(0) = 0$.*

Proof. The notation $\alpha(t) = o(t)$ means $\alpha(t) = t\beta(t)$, with $\beta(t) \to 0$ when $t \to 0$. Moreover β is F-smooth, since for all $u \in \mathcal{Y}_\circ$ the classical function β_u given by $\beta_u(t) = \langle \beta(t), u\rangle = \alpha_u/t$ is smooth. Next, consider the map $F : \mathbb{R}^2 \to \mathbb{C} : (s,t) \mapsto f(s\beta(t))$. If $c : \mathbb{R} \to \mathbb{R}^2$ is an arbitrary smooth curve, then $F \circ c$ is smooth. This implies that F is a smooth classical function. We have $\mathrm{D}(f \circ \alpha)(0) = \mathrm{D}_1F(0,0) + \mathrm{D}_2F(0,0) = 0$. □

Lemma 4.2. *If $f : \mathcal{Y} \to \mathbb{R}$, $\alpha, \beta : \mathbb{I} \to \mathcal{Y}$ are F-smooth, and $\alpha(0) = \beta(0) = 0$, then*

$$\mathrm{D}(f \circ (\alpha + \beta))(0) = \mathrm{D}(f \circ \alpha)(0) + \mathrm{D}(f \circ \beta)(0).$$

Proof. Define $F : \mathbb{R}^2 \to \mathbb{C} : (s,t) \mapsto f(\alpha(s) + \beta(t))$, a smooth classical function. We have $\mathrm{D}(f \circ (\alpha + \beta))(0) = \mathrm{D}_1F(0,0) + \mathrm{D}_2F(0,0) = \mathrm{D}(f \circ \alpha)(0) + \mathrm{D}(f \circ \beta)(0)$. □

Proposition 4.2. *If $\boldsymbol{N}$ is a classical manifold and $f : \mathcal{Y} \to \boldsymbol{N}$ is F-smooth, then there is a unique map $\mathrm{T}f : \mathrm{T}\mathcal{Y} \to \mathrm{T}\boldsymbol{N}$ over f, called the* tangent prolongation *of f, such that for any F-smooth curve α valued in $\mathcal{Y}$ one has*

$$\mathrm{T}f(\partial\alpha(t)) = \mathrm{T}(f \circ \alpha)(t).$$

Moreover, $\mathrm{T}f$ is linear over f and F-smooth.

Proof. Consider first the case $\boldsymbol{N} = \mathbb{R}$. Let $\alpha, \tilde{\alpha} : \mathbb{I} \to \mathcal{Y}$ be such that $\partial\alpha(0) = \partial\tilde{\alpha}(0)$. We have $\alpha(t) - \tilde{\alpha}(t) = o(t)$, hence

$$\mathrm{D}(f \circ \alpha)(0) - \mathrm{D}(f \circ \tilde{\alpha})(0) = \mathrm{D}(f \circ (\alpha - \tilde{\alpha}))(0) = 0,$$

namely the map $\mathrm{D}f : \mathrm{T}\mathcal{Y} \to \mathbb{R} : \partial\alpha(0) \mapsto \mathrm{D}(f \circ \alpha)(0)$ is well-defined. In order to see that $\mathrm{D}f$ is $\mathbb{R}$-linear over $\mathcal{Y}$, consider the curve $\beta(t) = \alpha(rt)$, fulfilling $\partial\beta(0) = r\partial\alpha(0)$. We obtain

$$\mathrm{D}f(r\partial\alpha(0)) = \mathrm{D}f(\partial\beta(0)) = \mathrm{D}(f \circ \beta)(0) = r\,\mathrm{D}(f \circ \alpha)(0) = r\,\mathrm{D}f(\partial\alpha(0))\,.$$

In order to see that $\mathrm{D}f$ is F-smooth, first note that, if $(\lambda, \mu) \in \mathcal{Y} \times \mathcal{Y} = \mathrm{T}\mathcal{Y}$, then $\mathrm{D}f(\lambda, \mu) = \mathrm{D}[f(\lambda + t\mu)](t = 0)$. If $(\phi, \psi) : \mathbb{R} \to \mathrm{T}\mathcal{Y}$ is any F-smooth curve, then consider the smooth classical function $F : \mathbb{R}^2 \to \mathbb{C} : (s, t) \mapsto f(\phi(s) + t\psi(s))$. We have

$$\mathrm{D}f(\phi(s), \psi(s)) = \mathrm{D}(f(\phi(s) + t\psi(s)))(t = 0) = \mathrm{D}_2F(s, 0)$$

which, as a partial derivative of a smooth function, depends smoothly on s. Finally, we set $\mathrm{T}f := (f \circ \mathrm{pr}_1\,, \mathrm{D}f) : \mathrm{T}\mathcal{Y} \to \mathbb{R} \times \mathbb{R}$.

If $\boldsymbol{N}$ is any classical manifold and (z^i) is a local coordinate chart on it, then by applying the above results to each component function $\mathsf{z}^i \circ f$ we complete the proof. □

Corollary 4.1. *Let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be an F-smooth map. Then there is a unique map*

$$\mathrm{T}\Phi \equiv (\Phi, \mathrm{D}\Phi) : \mathrm{T}\mathcal{Y} \to \mathrm{T}\mathcal{Z}\,,$$

called the tangent prolongation of Φ, such that $\mathrm{T}\Phi(\partial\alpha(t_0)) = \partial(\Phi \circ \alpha)(t_0)$ for any F-smooth curve α valued in $\mathcal{Y}$. Moreover, $\mathrm{T}\Phi$ is linear over Φ and F-smooth.

Proof. For each $u \in \mathcal{Z}_\circ$ we can apply proposition 4.1 to the map $\Phi_u : \mathcal{Y} \to \mathbb{C} : \lambda \mapsto \langle\Phi(\lambda), u\rangle$; setting

$$\langle \mathrm{D}\Phi(\partial\alpha(t_0)), u\rangle := \mathrm{D}\Phi_u(\partial\alpha(t_0)) = \mathrm{D}\langle\Phi \circ \alpha, u\rangle(t_0)$$

we obtain the stated result. □

It is not difficult to see that all tangent prolongations behave naturally in terms of any compositions.

For any two coordinate charts $\mathsf{y} : \breve{\boldsymbol{Y}} \to \boldsymbol{A} \subset \mathbb{R}^n$, $\mathsf{z} : \breve{\boldsymbol{Z}} \to \boldsymbol{B} \subset \mathbb{R}^n$, we obtain the induced charts $\mathsf{y}_* : \breve{\mathcal{Y}} \to \mathcal{R}^n$, $\mathsf{z}_* : \breve{\mathcal{Z}} \to \mathcal{R}^n$, and the 'coordinate expression' of $\mathrm{T}\Phi$ as

$$\mathrm{T}\Phi_{\mathsf{zy}} := \mathrm{T}(\mathsf{z}_* \circ \Phi \circ \mathsf{y}_*^{-1}) = \mathrm{T}\mathsf{z}_* \circ \mathrm{T}\Phi \circ \mathrm{T}(\mathsf{y}_*^{-1}) : \mathcal{A} \times \mathcal{R}^n \to \mathcal{B} \times \mathcal{R}^n\,.$$

In particular we may consider the transformation induced by a diffeomorphism $\varphi : \boldsymbol{Y} \to \boldsymbol{Z}$ (§2). There is a bijection between local F-smooth curves $\mathbb{R} \to \mathcal{Y}$ and $\mathbb{R} \to \mathcal{Z}$, given by $\alpha \mapsto \varphi_* \circ \alpha$; from $\langle\alpha\,, u\rangle = \langle\varphi_* \circ \alpha\,, \varphi_* u\rangle$, we see that φ_* is F-smooth. Since φ_* is linear, we obtain

$$\mathrm{T}\varphi_* = \varphi_* \times \varphi_*\,.$$

If $f : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{N}$ is F-smooth, where each one of the spaces $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{N}$ is either a classical manifold or a distribution space, then the *partial tangent prolongations*

$$\mathrm{T}_1 f : \mathrm{T}\mathcal{M}_1 \times \mathcal{M}_2 \to \mathrm{T}\mathcal{N}\,,$$
$$\mathrm{T}_2 f : \mathcal{M}_1 \times \mathrm{T}\mathcal{M}_2 \to \mathrm{T}\mathcal{N}\,,$$

are defined in a straightforward standard way, as well as the tangent prolongation

$$\mathrm{T}f : \mathrm{T}\mathcal{M}_1 \times \mathrm{T}\mathcal{M}_2 \to \mathrm{T}\mathcal{N}\,.$$

Similarly, if $f : \mathcal{M} \to \mathcal{N}_1 \times \mathcal{N}_2$, we have the tangent prolongation

$$\mathrm{T}f : \mathrm{T}\mathcal{M} \to \mathrm{T}(\mathcal{N}_1 \times \mathcal{N}_2) \cong \mathrm{T}\mathcal{N}_1 \times \mathrm{T}\mathcal{N}_2\,.$$

5. F-smooth bundles

By $\mathsf{p} : \boldsymbol{E} \to \boldsymbol{M}$ we shall denote a smooth classical bundle ($\dim \boldsymbol{M} = m$, $\dim \boldsymbol{E} = m{+}n$) whose fibres are smoothly oriented manifolds. This means that $\wedge^n \mathrm{V}\boldsymbol{E} \to \boldsymbol{E}$ is a trivializable bundle with smoothly oriented fibres, so that we also have the smooth 2-fibred bundle $\mathbb{V}\boldsymbol{E} \to \boldsymbol{E} \to \boldsymbol{M}$, where

$$\mathbb{V}\boldsymbol{E} := (\wedge^n \mathrm{V}\boldsymbol{E})^+ := \bigsqcup_{y\in \boldsymbol{E}} (\wedge^n \mathrm{V}_y \boldsymbol{E})^+ = \bigsqcup_{x\in \boldsymbol{M}} \mathbb{V}(\mathrm{T}\boldsymbol{E}_x)\,.$$

For each $x \in \boldsymbol{M}$ we consider the space $\mathcal{E}_x := \mathcal{U}'(\boldsymbol{E}_x\,,\, \mathbb{C}\otimes \mathbb{V}^{-1/2}\boldsymbol{E}_x)$ of all complex generalized half densities on $\boldsymbol{E}_x$. Next we introduce the fibred set

$$\wp : \mathcal{E} := \bigsqcup_{x\in \boldsymbol{M}} \mathcal{E}_x \to \boldsymbol{M}\,.$$

Let $\boldsymbol{X} \subset \boldsymbol{M}$ be an open submanifold. A local bundle trivialization $(\mathsf{x},\mathsf{y}) : \boldsymbol{E}_{\boldsymbol{X}} \to \boldsymbol{X} \times \boldsymbol{Y}$ yields the local bundle trivialization

$$(\mathsf{x},\mathsf{y}_*) : \mathcal{E}_{\boldsymbol{X}} \to \boldsymbol{X} \times \mathcal{Y} : \lambda_x \mapsto \big(x, (\mathsf{y}_x)_*\lambda_x\big)\,,$$

where $\lambda_x \in \mathcal{E}_x$ and y_x is the restriction of y to $\boldsymbol{E}_x$. If (x,y) and $(\mathsf{x},\mathsf{z}) : \boldsymbol{E}_{\boldsymbol{X}} \to \boldsymbol{X} \times \boldsymbol{Z}$ are smooth trivializations (for simplicity we take the base chart x to be the same), then $(\mathsf{x},\mathsf{z}_*) \circ (\mathsf{x},\mathsf{y}_*)^{-1} : \boldsymbol{X} \times \mathcal{Y} \to \boldsymbol{X} \times \mathcal{Z}$ is F-smooth. This implies that a bundle atlas of $\boldsymbol{E}$ yields an F-smooth bundle atlas of $\mathcal{E}$. We also have the subbundles $\mathcal{E}_\circ \subset \mathcal{H} \subset \mathcal{E}$, which behave naturally in terms of bundle trivializations.

Clearly, $\mathcal{E}$ turns out to be an F-smooth space in a natural way: a curve $\alpha : \mathbb{I} \to \mathcal{E}$ is defined to be F-smooth if $(\mathsf{x},\mathsf{y}_*) \circ \alpha$ is such for any bundle trivialization (x,y); in other terms, $\mathsf{x} \circ \alpha$ is a classical smooth curve and $\mathsf{y}_* \circ \alpha$ is an F-smooth curve in the sense of §3. In general, the F-smoothness of a map can be expressed via its trivialized expression. So, $f : \mathcal{E} \to \mathbb{R}$ is F-smooth iff $f \circ (\mathsf{x},\mathsf{y}_*)^{-1}$ is such for every (x,y), as one sees from

$$f \circ \alpha = f \circ (\mathsf{x},\mathsf{y}_*)^{-1} \circ (\mathsf{x},\mathsf{y}_*) \circ \alpha\,.$$

Similarly, a map $\phi : \boldsymbol{N} \to \mathcal{E}$, where $\boldsymbol{N}$ is a classical manifold, is F-smooth iff $(\mathsf{x},\mathsf{y}_*) \circ \phi$ is such for every (x,y). Again another similar statement holds for a morphism between F-smooth bundles.

Let $\mathfrak{C}_\mathcal{E}$ be the set of all F-smooth curves in $\mathcal{E}$; if $\alpha \in \mathfrak{C}_\mathcal{E}$ then $\mathrm{T}((\mathsf{x},\mathsf{y}_*) \circ \alpha) : \mathbb{I} \times \mathbb{R} \to \mathrm{T}\boldsymbol{X} \times \mathrm{T}\mathcal{Y}$ is naturally defined as

$$\mathrm{T}((\mathsf{x},\mathsf{y}_*) \circ \alpha) = \Big(\mathrm{T}(\mathsf{x} \circ \alpha)\,,\, \mathrm{T}(\mathsf{y}_* \circ \alpha)\Big)\,.$$

We say that two such curves are first-order equivalent if their trivialized expressions are such; in this way we obtain the definition of the tangent space $\mathrm{T}\mathcal{E}$. Obviously this is a fibred set over $\mathcal{E}$; a local bundle trivialization (x,y) on $\boldsymbol{E}$ yields the local bundle trivialization

$$\mathrm{T}(\mathsf{x},\mathsf{y}_*):\mathrm{T}\mathcal{E}\to\mathrm{T}\boldsymbol{X}\times\mathrm{T}\mathcal{Y}\,,$$

and the transition maps between two induced trivializations are F-smooth and linear. Hence a smooth atlas of $\boldsymbol{E}$ yields an F-smooth atlas of $\mathrm{T}\mathcal{E}$, so that $\pi_{\mathcal{E}}:\mathrm{T}\mathcal{E}\to\mathcal{E}$, the tangent bundle of $\mathcal{E}$, is an F-smooth vector bundle. We have another F-smooth bundle with the same total F-smooth space, namely

$$\mathrm{T}\wp:\mathrm{T}\mathcal{E}\to\mathrm{T}\boldsymbol{M}:\partial\alpha\mapsto\partial(\mathsf{p}\circ\alpha)\,.$$

Moreover we have the *vertical subbundle* over $\mathcal{E}$

$$\mathrm{V}\mathcal{E}:=\operatorname{Ker}\mathrm{T}\wp\subset\mathrm{T}\mathcal{E}\,;$$

since $(\mathrm{V}\mathcal{E})_x=\mathrm{T}(\mathcal{E}_x)=\mathcal{E}_x\times\mathcal{E}_x$, we also have

$$\mathrm{V}\mathcal{E}=\mathcal{E}\underset{\boldsymbol{M}}{\times}\mathcal{E}\,.$$

Summarizing, we have the exact sequence over $\mathcal{E}$

$$0\to\mathrm{V}\mathcal{E}\to\mathrm{T}\mathcal{E}\to\mathcal{E}\underset{\boldsymbol{M}}{\times}\mathrm{T}\boldsymbol{M}\to 0\,.$$

The subbundle of $\mathrm{T}^*\boldsymbol{M}\otimes_{\mathcal{E}}\mathrm{T}\mathcal{E}$ which projects over the identity of $\mathrm{T}\boldsymbol{M}$ is called the *first jet bundle*, denoted by $\mathrm{J}\mathcal{E}\to\mathcal{E}$. This is an affine bundle over $\mathcal{E}$, with 'derived' vector bundle $\mathrm{T}^*\boldsymbol{M}\otimes_{\mathcal{E}}\mathrm{V}\mathcal{E}$. The restriction of $\mathrm{T}^*\mathsf{x}\otimes\mathrm{T}(\mathsf{x},\mathsf{y}_*)$ is a local bundle trivialization which we denote by

$$\mathrm{J}(\mathsf{x},\mathsf{y}_*):\mathrm{J}\mathcal{E}\to\mathrm{J}(\boldsymbol{X}\times\mathcal{Y})\cong\mathcal{Y}\times\Big(\mathrm{T}^*\boldsymbol{X}\otimes\mathcal{Y}\Big)\,.$$

Replacing the base map x by a coordinate chart $\mathsf{x}=(\mathsf{x}^a)$ we have the fibred charts

$$\begin{aligned}
&(\mathsf{x},\mathsf{y}_*):\mathcal{E}\to\mathbb{R}^m\times\mathcal{Y}\,,\\
&(\mathsf{x}^a,\mathsf{y}_*\,,\dot{\mathsf{x}}^a,\dot{\mathsf{y}}_*):=\mathrm{T}(\mathsf{x},\mathsf{y}_*):\mathrm{T}\mathcal{E}\to\mathbb{R}^m\times\mathcal{Y}\times\mathbb{R}^m\times\mathcal{Y}\,,\\
&(\mathsf{x}^a,\mathsf{y}_*\,,\mathsf{y}_{*a}):=\mathrm{J}(\mathsf{x},\mathsf{y}_*):\mathrm{J}\mathcal{E}\to\mathbb{R}^m\times\mathcal{Y}\times(\mathbb{R}^m\otimes\mathcal{Y})\,.
\end{aligned}$$

Possibly we may have $\boldsymbol{Y}\subset\mathbb{R}^n$ so that $\mathsf{y}=(\mathsf{y}^i)$ is a set of fibre coordinates; then $\mathcal{Y}\subset\mathcal{R}^n$.

Tangent prolongations of F-smooth maps $\boldsymbol{N}\to\mathcal{E}$ and $\mathcal{E}\to\boldsymbol{N}$, where $\boldsymbol{N}$ is a classical manifold, and of F-smooth maps between distributional bundles, are easily defined in terms of the tangent prolongations of their local trivialized (chart) expressions; all turn out to be F-smooth linear morphisms.

In particular, if $\sigma:\boldsymbol{M}\to\mathcal{E}$ is an F-smooth section, then $\mathrm{T}s:\mathrm{T}\boldsymbol{M}\to\mathrm{T}\mathcal{E}$ projects over the identity of $\mathrm{T}\boldsymbol{M}$, so that it can be viewed as a section $\mathrm{j}\sigma:\boldsymbol{M}\to\mathrm{J}\mathcal{E}$. Let $\sigma_{\mathsf{y}}:=\mathsf{y}_*\circ\sigma:\boldsymbol{M}\to\mathcal{Y}$ be the 'coordinate expression' of σ. Then the coordinate expressions of $\mathrm{T}\sigma$ and $\mathrm{j}\sigma$ are

$$\begin{aligned}
&(\mathsf{x}^a,\mathsf{y}_*\,,\dot{\mathsf{x}}^a,\dot{\mathsf{y}}_*)\circ\mathrm{T}\sigma=\mathrm{T}\sigma_{\mathsf{y}}=(\mathsf{x}^a,\sigma_{\mathsf{y}},\dot{\mathsf{x}}^a,\dot{\mathsf{x}}^a\partial_a\sigma_{\mathsf{y}})\,,\\
&(\mathsf{x}^a,\mathsf{y}_*\,,\mathsf{y}_{*a})\circ\mathrm{j}\sigma=\mathrm{J}\sigma_{\mathsf{y}}=(\mathsf{x}^a,\sigma_{\mathsf{y}}\,,\partial_a\sigma_{\mathsf{y}})\,.
\end{aligned}$$

For maps $f:\mathcal{E}\to\mathbb{R}$ we introduce the notation

$$\frac{\partial f}{\partial \mathsf{y}_*} := [\mathrm{D}_2(f\circ(\mathsf{x},\mathsf{y}_*)^{-1})]\circ(\mathsf{x},\mathsf{y}_*):\mathcal{E}\times\mathcal{Y}\to\mathbb{R}\,,$$

which plays the role which, in classical bundles, is of the set of partial derivatives with respect to fibre coordinates. We obtain the local coordinate expression

$$\mathrm{d}f := \mathrm{pr}_1\circ\mathrm{T}f = \partial_a f\,\mathrm{d}\mathsf{x}^a + \frac{\partial f}{\partial\mathsf{y}_*}\circ\mathrm{d}\mathsf{y}_*\,.$$

More generally, if $f:\mathcal{E}\to\boldsymbol{N}$ and (v^k) is a coordinate chart on $\boldsymbol{N}$, then we obtain the local expression

$$\dot{\mathsf{v}}^k\circ\mathrm{T}f = \partial_a\mathsf{v}^k\,\dot{\mathsf{x}}^a + \frac{\partial\mathsf{v}^k}{\partial\mathsf{y}_*}\circ\dot{\mathsf{y}}_*\,.$$

Let $\boldsymbol{F}\to\boldsymbol{M}$ be another bundle over the same base manifold $\boldsymbol{M}$, and $\Phi:\mathcal{E}\to\mathcal{F}$ a fibred F-smooth morphism over $\boldsymbol{M}$ (we assume, for simplicity, that Φ is constant on the base). Then the fibred morphisms

$$\mathrm{T}\Phi:\mathrm{T}\mathcal{E}\to\mathrm{T}\mathcal{F}\,,\quad \mathrm{J}\Phi:\mathrm{J}\mathcal{E}\to\mathrm{J}\mathcal{F}$$

are characterized by

$$\mathrm{T}\Phi\circ\mathrm{T}\sigma = \mathrm{T}(\Phi\circ\sigma)\,,\quad \mathrm{J}\Phi\circ\mathrm{j}\sigma = \mathrm{j}(\Phi\circ\sigma)$$

holding for every F-smooth section $\sigma:\boldsymbol{M}\to\mathcal{E}$. Let $(\mathsf{x},\mathsf{z}):\boldsymbol{F}\to\mathbb{R}^m\times\boldsymbol{Z}$ be a local chart; setting $\Phi_{\mathsf{zy}} := \mathsf{z}_*\circ\Phi\circ(\mathsf{x},\mathsf{y}_*)^{-1}$ and

$$\frac{\partial\Phi_{\mathsf{z}}}{\partial\mathsf{y}_*} := (\mathrm{D}_2(\Phi_{\mathsf{zy}}))\circ(\mathsf{x},\mathsf{y}):\mathcal{E}\times\mathcal{Y}\to\mathcal{Z}\,,$$

$$\text{i.e.}\quad \frac{\partial\Phi_{\mathsf{z}}}{\partial\mathsf{y}_*}(\lambda) = \mathrm{D}(\mathsf{z}_*\circ\Phi_x\circ\mathsf{y}_*^{-1})(\lambda_{\mathsf{y}}):\mathcal{Y}\to\mathcal{Z}\,,\quad x:=\mathsf{p}(\lambda)\,,$$

we obtain the local expressions

$$\dot{\mathsf{z}}_*\circ\mathrm{T}\Phi = \partial_a\Phi_{\mathsf{z}}\,\dot{\mathsf{x}}^a + \frac{\partial\Phi_{\mathsf{z}}}{\partial\mathsf{y}_*}\circ\dot{\mathsf{y}}_*\,,$$

$$\mathsf{z}_{*a}\circ\mathrm{J}\Phi = \partial_a\Phi_{\mathsf{z}} + \frac{\partial\Phi_{\mathsf{z}}}{\partial\mathsf{y}_*}\circ\mathsf{y}_{*a}\,.$$

In particular, let $\varphi:\boldsymbol{E}\to\boldsymbol{F}$ be a fibred diffeomorphism over $\boldsymbol{M}$ and $\Phi\equiv\varphi_*$. Then we obtain (the second line follows from the linearity of φ_*)

$$\Phi_{\mathsf{z}}(\lambda) = (\lambda_{\mathsf{y}}\circ\varphi_x^{-1})\sqrt{|\varphi_x^{-1}|_{\mathsf{yz}}}\,,\quad x:=\mathsf{p}(\lambda)\,,$$

$$\frac{\partial\Phi_{\mathsf{z}}}{\partial\mathsf{y}_*} = \Phi_{\mathsf{zy}}\circ\mathsf{x}\,,$$

$$\partial_a\Phi_{\mathsf{z}}(\lambda) = \mathrm{T}_1\Phi_{\mathsf{z}}(\partial\mathsf{x}_a\,,\lambda) =$$
$$= \big[(\partial_i\lambda_{\mathsf{y}}\circ\varphi^{-1})\partial_a(\varphi^{-1})^i + \tfrac{1}{2}(\lambda_{\mathsf{y}}\circ\varphi^{-1})\big((\partial_i\varphi^j\circ\varphi^{-1})\partial_a\partial_j(\varphi^{-1})^i\big)\big]\sqrt{|\varphi_x^{-1}|_{\mathsf{yz}}}\,.$$

If Φ is the identity map of $\mathcal{E}$, then we obtain the transition maps under fibred coordinate transformations (with fixed base coordinates) respectively on $\mathrm{T}\mathcal{E}$ and $\mathrm{J}\mathcal{E}$. We set

$$\begin{aligned} &\kappa_{\mathsf{zy}} := [\mathsf{z} \circ (\mathsf{x}, \mathsf{y})^{-1}] \circ \mathsf{x} : \boldsymbol{E} \times \boldsymbol{Y} \to \boldsymbol{Z} \,, \\ &K_{\mathsf{zy}} := (\kappa_{\mathsf{zy}})_* = [\mathsf{z}_* \circ (\mathsf{x}, \mathsf{y}_*)^{-1}] \circ \mathsf{x} : \mathcal{E} \times \mathcal{Y} \to \mathcal{Z} \,, \\ \Rightarrow\quad &\lambda_{\mathsf{z}} = K_{\mathsf{zy}} \lambda_{\mathsf{y}} = (\lambda_{\mathsf{y}} \circ \kappa_{\mathsf{yz}}) \sqrt{|\kappa_{\mathsf{yz}}|} \end{aligned}$$

(note that $\kappa_{\mathsf{yz}} = (\kappa_{\mathsf{zy}})^{-1}$) and obtain

$$\begin{aligned} \dot{\mathsf{z}}_* &= \dot{\mathsf{x}}^a \partial_a K_{\mathsf{zy}} \circ \mathsf{y}_* + K_{\mathsf{zy}} \circ \dot{\mathsf{y}}_* = \\ &= \Big(\dot{\mathsf{x}}^a \big[(\partial_i \mathsf{y}_* \circ \kappa_{\mathsf{yz}}) \partial_a (\kappa_{\mathsf{yz}})^i + \tfrac{1}{2} (\mathsf{y}_* \circ \kappa_{\mathsf{yz}}) (\partial_i (\kappa_{\mathsf{zy}})^j \partial_a \partial_j (\kappa_{\mathsf{yz}})^i) \big] + (\dot{\mathsf{y}}_* \circ \kappa_{\mathsf{yz}}) \Big) \sqrt{|\kappa_{\mathsf{yz}}|} \,, \end{aligned}$$

$$\begin{aligned} \mathsf{z}_{*a} &= \partial_a K_{\mathsf{zy}} \circ \mathsf{y}_* + K_{\mathsf{zy}} \circ \mathsf{y}_{*a} = \\ &= \Big(\big[(\partial_i \mathsf{y}_* \circ \kappa_{\mathsf{yz}}) \partial_a (\kappa_{\mathsf{yz}})^i + \tfrac{1}{2} (\mathsf{y}_* \circ \kappa_{\mathsf{yz}}) (\partial_i (\kappa_{\mathsf{zy}})^j \partial_a \partial_j (\kappa_{\mathsf{yz}})^i) \big] + (\mathsf{y}_{*a} \circ \kappa_{\mathsf{yz}}) \Big) \sqrt{|\kappa_{\mathsf{yz}}|} \,, \end{aligned}$$

where $\partial_i \mathsf{y}_* : \lambda \mapsto \partial_i \lambda_{\mathsf{y}}$.

6. F-smooth connections

As in the standard finite-dimensional case, a *connection* on the distributional bundle $\mathcal{E}$ is defined to be an F-smooth section

$$\Gamma : \mathcal{E} \to \mathrm{J}\mathcal{E} \,.$$

In the domain of a fibred coordinate chart $(\mathsf{x}^a, \mathsf{y}^i)$ we have the local expression

$$\Gamma_{a\mathsf{y}} := \mathsf{y}_{*a} \circ \Gamma : \mathcal{E} \to \mathcal{R}^n \,.$$

We shall only consider *linear* connections, that is connections Γ which are linear morphisms over $\boldsymbol{M}$. Then we write $\Gamma_{a\mathsf{y}} = \Gamma_{a\mathsf{yy}} \circ \mathsf{y}_*$ where $\Gamma_{a\mathsf{yy}} : \boldsymbol{X} \times \mathcal{R}^n \to \mathcal{R}^n$ or also

$$\Gamma_{a\mathsf{yy}} : \boldsymbol{X} \to \mathcal{O}(\mathcal{R}^n) \,,$$

where $\mathcal{O}(\mathcal{R}^n)$ denotes the vector space of all linear operators in $\mathcal{R}^n$. The existence of global connections then follows from standard arguments using the paracompactness of $\boldsymbol{M}$.

If $\Gamma_{a\mathsf{yy}}$ and $\Gamma_{a\mathsf{zz}}$ are the local expressions of Γ in two different fibred charts (x, y) and (x, z), then we have

$$\Gamma_{a\mathsf{zz}} = (\partial_a K_{\mathsf{zy}} + K_{\mathsf{zy}} \circ \Gamma_{a\mathsf{yy}}) \circ K_{\mathsf{yz}} \,.$$

As in the finite-dimensional case, a connection yields a number of structures (whose assignment is actually equivalent to that of the connection itself). First, Γ can be viewed as a linear map $\mathcal{E} \times_{\boldsymbol{M}} \mathrm{T}\boldsymbol{M} \to \mathrm{T}\mathcal{E}$, and $(\pi_{\mathcal{E}} \,, \mathrm{T}\wp) \circ \Gamma$ is the identity of $\mathcal{E} \times_{\boldsymbol{M}} \mathrm{T}\boldsymbol{M}$. The image

$$\mathrm{H}_\Gamma \mathcal{E} := \Gamma(\mathcal{E} \underset{\boldsymbol{M}}{\times} \mathrm{T}\boldsymbol{M})$$

is a vector subbundle of $\mathrm{T}\mathcal{E} \to \mathcal{E}$, with m-dimensional fibres; the restriction of $\Gamma \circ (\pi_{\mathcal{E}} \,, \mathrm{T}\wp)$ is the identity of $\mathrm{H}_\Gamma \mathcal{E}$. If $v : \boldsymbol{M} \to \mathrm{T}\boldsymbol{M}$ is a smooth vector field, then

$\Gamma_v : \mathcal{E} \to \mathrm{T}\mathcal{E}$ is an F-smooth vector field, called its *horizontal lift*, with coordinate expression

$$\dot{\mathsf{x}}^a \circ \Gamma_v = v^a \,, \quad \dot{\mathsf{y}}_* \circ \Gamma_v = v^a \Gamma_{a\mathsf{y}} \,.$$

We also have the complementary map

$$\Omega := 1\!\!1 - \Gamma : \mathrm{T}\mathcal{E} \to \mathrm{V}\mathcal{E}$$

(it is immediate to check that Ω is vertical valued) so that the map $\big(\Gamma \circ (\pi_{\mathcal{E}} \,, \mathrm{T}\wp) \,, \Omega\big)$ determines the decomposition

$$\mathrm{T}\mathcal{E} = \mathrm{H}_\Gamma \mathcal{E} \underset{\mathcal{E}}{\oplus} \mathrm{V}\mathcal{E} \,.$$

Let $\sigma : \boldsymbol{M} \to \mathcal{E}$ be an F-smooth section. The *covariant derivative* of σ is defined to be the linear morphsim over $\boldsymbol{M}$

$$\nabla\sigma := \mathrm{pr}_2 \circ \Omega \circ \mathrm{T}\sigma : \mathrm{T}\boldsymbol{M} \to \mathcal{E} \,.$$

If $v : \boldsymbol{M} \to \mathrm{T}\boldsymbol{M}$ is a vector field we also write $\nabla_v \sigma := \nabla\sigma \circ v$. The local coordinate expression of the covariant derivative is

$$(\nabla\sigma)_{\mathsf{y}} := \mathsf{y}_* \circ \nabla\sigma = \dot{\mathsf{x}}^a \partial_a \sigma_{\mathsf{y}} - \Gamma_{a\mathsf{y}} \circ \sigma \,.$$

We want to show that a classical smooth connection $\gamma : \boldsymbol{E} \to \mathrm{J}\boldsymbol{E}$ yields a connection on $\mathcal{E} \to \boldsymbol{M}$ in a natural way. We first perform a preliminary construction.

Let $\eta : \boldsymbol{E} \to \wedge^n \mathrm{V}^*\boldsymbol{E}$ be a smooth section. Denote by $\omega : \mathrm{T}\boldsymbol{E} \to \mathrm{V}\boldsymbol{E}$ the vertical projection associated with γ. Then we have

$$\begin{aligned} &\omega^*\eta : \boldsymbol{E} \to \wedge^n \mathrm{T}^*\boldsymbol{E} \,, \\ &\mathrm{d}(\omega^*\eta) : \boldsymbol{E} \to \wedge^{n+1} \mathrm{T}^*\boldsymbol{E} \,, \\ &\gamma \lrcorner \mathrm{d}(\omega^*\eta) : \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \mathrm{T}\boldsymbol{M} \to \wedge^n \mathrm{T}^*\boldsymbol{E} \end{aligned}$$

Denoting by $\iota : \mathrm{V}\boldsymbol{E} \to \mathrm{T}\boldsymbol{E}$ the standard inclusion, we set

$$\nabla\eta := \iota^* \circ [\gamma \lrcorner \mathrm{d}(\omega^*\eta)] : \boldsymbol{E} \underset{\boldsymbol{M}}{\times} \mathrm{T}\boldsymbol{M} \to \wedge^n \mathrm{V}^*\boldsymbol{E} \,.$$

Let now $\sigma : \boldsymbol{M} \to \mathcal{E}$ be a section corresponding to an ordinary smooth section $\boldsymbol{E} \to \mathbb{C} \otimes \mathbb{V}^{-1/2}\boldsymbol{E}$ (defined on a 'tubelike' open submanifold of $\boldsymbol{E}$). If σ nowhere vanishes we set

$$\nabla\sigma := \frac{1}{2\sigma} \nabla(\sigma^2) \,,$$

which has the local coordinate expression

$$(\nabla\sigma)_{\mathsf{y}} = \partial_a \sigma_{\mathsf{y}} + \gamma^i_a \partial_i \sigma_{\mathsf{y}} + \tfrac{1}{2}(\partial_i \gamma^i_a) \sigma_{\mathsf{y}}$$

(more properly, here we should write $\gamma^i_a \circ \mathsf{y}^{-1} : \boldsymbol{X} \times \mathbb{R}^n \to \mathbb{R}$). From this it is clear that $\nabla\sigma$ can be extended by continuity to all sections $\boldsymbol{M} \to \mathcal{E}$; we thus have an induced connection Γ on $\mathcal{E} \to \boldsymbol{M}$. Note that Γ is linear, even if γ is not. Its local coordinate expression is

$$\Gamma_{a\mathsf{y}\mathsf{y}} = -\gamma^i_a \partial_i - \tfrac{1}{2}(\partial_i \gamma^i_a) 1\!\!1_{\mathcal{Y}} \,.$$

It is not difficult to check that, under a change of the fibre coordinates, $\Gamma_{a\mathsf{y}\mathsf{y}}$ transforms in the right way for the components of a connection.

The connection Γ induced by γ has a simple geometric interpretation. Namely, consider a smooth curve $c : \mathbb{I} \to \boldsymbol{M}$. Suppose that, for sufficiently close $t, t_0 \in \mathbb{I}$, γ yields a diffeomorphism $\varphi_t : \boldsymbol{E}_{c(t_0)} \to \boldsymbol{E}_{c(t)}$ via parallel transport along c (this is certainly the case if $\boldsymbol{E}$ is a vector bundle and γ is linear). Then, for each $\lambda \in \mathcal{E}_{c(t_0)}$ the smooth curve

$$\mathbb{I} \to \mathcal{E} : t \mapsto (\varphi_t)_* \lambda$$

is exactly the *horizontal lift* of c (in $\mathcal{E}$) through λ. For a general classical connection, the map φ_t will be not defined on the whole fibre $\mathcal{E}_{c(t_0)}$ (even for t arbitrarily close to t_0), but the above interpretation applies, for example, whenever λ has compact support.

7. Brackets and curvature

This section is a brief summary of statements which either can be developped in strict analogy with the finite-dimensional situation, or follow from direct calculations.

Let $v, w : \mathcal{E} \to \mathrm{T}\mathcal{E}$ be F-smooth vector fields, so that $\mathrm{T}v, \mathrm{T}w : \mathrm{T}\mathcal{E} \to \mathrm{TT}\mathcal{E}$. We have a canonical involution $\mathrm{s} : \mathrm{TT}\mathcal{E} \to \mathrm{TT}\mathcal{E}$, and

$$\mathrm{T}w \circ v - \mathrm{s}(\mathrm{T}v \circ w) : \mathcal{E} \to \mathrm{VT}\mathcal{E} \cong \mathrm{T}\mathcal{E} \underset{\mathcal{E}}{\times} \mathrm{T}\mathcal{E} \,.$$

Then we define the Lie bracket of the two vector fields to be

$$[v, w] := \mathrm{pr}_2\big(\mathrm{T}w \circ v - \mathrm{s}(\mathrm{T}v \circ w)\big) : \mathcal{E} \to \mathrm{T}\mathcal{E} \,,$$

which has the local expression

$$\begin{aligned} [v, w]^a &= v^b \partial_b w^a - w^b \partial_b v^a + \partial_{\mathsf{y}*} w^a \circ v_{\mathsf{y}} - \partial_{\mathsf{y}*} v^a \circ w_{\mathsf{y}} \,, \\ [v, w]_{\mathsf{y}} &= v^b \partial_b w_{\mathsf{y}} - w^b \partial_b v_{\mathsf{y}} + \partial_{\mathsf{y}*} w_{\mathsf{y}} \circ v_{\mathsf{y}} - \partial_{\mathsf{y}*} v_{\mathsf{y}} \circ w_{\mathsf{y}} \end{aligned}$$

The Frölicher-Nijenhuis bracket of tangent-valued forms can be introduced by a straightforward extension of the standard definition, which is given in terms of the Lie bracket of vector fields [FN56, MK98, MM84, KMS93]. In particular we are interested in 'basic' forms $\mathcal{E} \to \wedge \mathrm{T}^*\boldsymbol{M} \otimes_M \mathrm{T}\mathcal{E}$. If Γ is a connection on $\mathcal{E}$ then its curvature is defined to be

$$R := \tfrac{1}{2}[\Gamma, \Gamma] : \mathcal{E} \to \wedge^2 \mathrm{T}^*\boldsymbol{M} \underset{\mathcal{E}}{\otimes} \mathrm{V}\mathcal{E} \,,$$

which has the coordinate expression

$$R_{\mathsf{yy}} = R_{ab\mathsf{yy}} \mathrm{dx}^a \wedge \mathrm{dx}^b = (\partial_a \Gamma_{b\mathsf{yy}} + \Gamma_{b\mathsf{yy}} \circ \Gamma_{a\mathsf{yy}}) \mathrm{dx}^a \wedge \mathrm{dx}^b \,.$$

In particular, if Γ is the connection determined by the classical connection γ, then we find

$$R_{ab\mathsf{yy}} = -\rho_{ab}{}^i \partial_i - \tfrac{1}{2}(\partial_i \rho_{ab}{}^i) \mathbb{1}_{\mathcal{Y}} \,,$$

where ρ denotes the classical curvature of γ.

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